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A Proof of the Equivalence Between the Polytabloid Bases and Specht Polynomials for Irreducible Representations of the Symmetric Group

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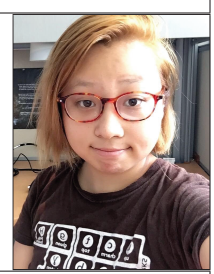

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Abstract We show an equivalence between the standard basis vectors in the space of polytabloids and the basis vectors of Specht polynomials for irreducible representations of the symmetric group. The authors have been unable to find any published proof of this result.

1 Introduction

Given a partition λ of an integer n, there is a way to construct an irreducible representation of the symmetric group S_n, and all irreducible representations of S_n can be constructed in this way. Thus the irreducible representations of the symmetric groups are in bijective correspondence with partitions of integers.

There are multiple ways to realize the irreducible representations of S_n. One of these ways is as a certain vector space consisting of complex polynomials $\mathcal{P}^\lambda$, and another is as a vector space S^λ of polytabloids. In this paper we will briefly define these representations and then prove that they are equivalent.

The representation theory of the symmetric group S_n provides much intuition into more advanced concepts in the field, and so it is fruitful to understand these representations from multiple perspectives.

2 Symmetric group, polynomials and Young tableaux

Let S_n be the *symmetric group* on n letters, that is, the group of bijections on the set $\{1, 2, \ldots, n\}$ acting on the left, so that for $\sigma, \tau \in S_n$, $(\sigma \cdot \tau)(i) = \sigma(\tau(i))$. Elements of S_n are typically written using cycle notation. For example, if $\sigma \in S_5$, and if

$$\sigma(1) = 3, \sigma(3) = 1, \sigma(2) = 4, \sigma(4) = 5, \text{ and } \sigma(5) = 2,$$

we write $\sigma = (1,3)(2,4,5)$. The identity element is written as (1), and it is a standard result that any element in S_n can be written as a product of two-cycles; for example, $(1,2,3,4) = (1,2)(2,3)(3,4)$. It is also a standard result that disjoint cycles commute; for example $(1,2,3)(4,5) = (4,5)(1,2,3)$.

Given any finite group such as S_n we can form the *group algebra* $\mathbb{C}[S_n]$ by first declaring that the group elements are a basis for a $\mathbb{C}$-vector space, then extending the group multiplication via the distributive property over addition to give $\mathbb{C}[S_n]$ an algebraic structure. For example,

$$[2(1,2) + 3(2,3)] \times [(1,3) + 2(1,3,2)] = 2(1,3,2) + 3(1,2,3) + 4(1,3) + 6(1,2).$$

Let $\mathcal{P}(x_1, \ldots, x_n) = \mathcal{P}$ be the vector space of polynomials in n variables with complex coefficients. The group S_n acts on this space by permuting the variables and then extending by linearity, and hence we have a representation of S_n on $\mathcal{P}$. We use the standard "lower dot" notation for a group representation,

$$\sigma.x_i = x_{\sigma(i)}.$$

For example,

$$(1,2,3).(x_1 x_2^2 x_3^3 + 2x_1^2 x_3) = x_2 x_3^2 x_1^3 + 2x_2^2 x_1.$$

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$ be a *partition* of n, which means that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n, \quad \text{with} \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0.$$

Equivalently, a partition λ can be depicted as a *Young diagram*, which is a left and top-justified array of ℓ rows of boxes where the ith row has length λ_i. It is customary to use λ to denote the partition or its associated Young diagram interchangeably.

A *Young tableau* t is a Young diagram where the boxes are filled with elements from an ordered set, sometimes called an *alphabet*. Typically the entries are positive integers, but we also see variables such as $\{x_i\}$ ordered by their subscripts. A *standard Young tableau* is a Young tableau where the entries are taken without replacement from $\{1, 2, ..., n\}$, and where the entries across each row and down each column are increasing.

A *tabloid* $\{t\}$ is an equivalence class of Young tableaux of the same shape represented by tableau t, where two tableaux (which need not be standard) are declared equivalent if each corresponding row contains the same entries.

Example 1. *Let $\lambda = (4, 2, 2, 1)$ be a partition of 9. Then*

$$\lambda = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & \\ \cline{1-2} & \\ \cline{1-2} \\ \cline{1-1} \end{array} \qquad \textit{is the associated Young diagram.}$$

$$t = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 \\ \cline{1-2} 6 & 8 \\ \cline{1-2} 9 \\ \cline{1-1} \end{array} \qquad \textit{is a standard Young tableau.}$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 5 & 3 & 7 \\ \hline 2 & 4 \\ \cline{1-2} 8 & 6 \\ \cline{1-2} 9 \\ \cline{1-1} \end{array} \qquad \textit{is equivalent to } t.$$

We denote the associated tabloid by deleting the vertical lines. For this example,

$$\{t\} = \begin{array}{cccc} \hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 \\ \cline{1-2} 6 & 8 \\ \cline{1-2} 9 \\ \cline{1-1} \end{array} .$$

Let $\mathcal{T}$ denote the set of all tabloids, and let $\mathcal{T}^\lambda$ denote the subset of all tabloids with shape λ. By declaring the set of tabloids in $\mathcal{T}^\lambda$ to be basis vectors, we construct $\mathbb{C}[\mathcal{T}^\lambda]$, the $\mathbb{C}$-vector space of *polytabloids of shape* λ. The symmetric group S_n acts on a Young tableau (and its associated tabloid) by permuting the entries, and thus – extending this action by linearity – the vector space $\mathbb{C}[\mathcal{T}^\lambda]$ is also a representation of S_n.

3 Representations

Definition 2. A representation of a group G on a vector space V is a homomorphism $\alpha : G \to GL(V)$

Definition 3. Let α be a representation of a group G on a vector space V and let W be a suspace of V. If $\alpha(g)w \in W$ for all $w \in W$ and $g \in G$ then W is a G–invariant subspace of V.

Definition 4. A representation of a group G on a vector space V is irreducible if V contains no proper nonzero G–invariant subspaces.

Example 5. *Consider the representation of the group S_3 on the vector space $\mathbb{R}^3$, where the homomorphism $\alpha : S_3 \to GL(\mathbb{R}^3)$ permutes the standard basis vectors. Then the subspaces $I := \{(a,a,a)|a \in R\}$ and $W := \{(a,b,c)|a+b+c = 0\}$ are G-invariant subspaces that are irreducible.*

4 The Polytabloid and Specht bases

Let t be a standard Young tableau, let V_t be the subgroup of S_n that preserves each column of t, and define the *column anti-symmetrizer* by

$$\mathcal{V}_t := \sum_{\sigma \in V_t} \operatorname{sgn}(\sigma)\,\sigma,$$

which is an element of the group algebra $\mathbb{C}[S_n]$. Finally, set $e_t := \mathcal{V}_t.\{t\}$, a vector in $\mathbb{C}[\mathcal{T}^\lambda]$.

By Theorem 2.5.2 from [S] or Theorem 7.2.7 from [K-J], the set

$$\{e_t : t \text{ is a standard } \lambda\text{-tableau}\}$$

is a basis for an irreducible representation of S_n in $\mathbb{C}[\mathcal{T}]$, denoted $\mathcal{S}^\lambda$ and called the *Specht module*.

Example 6. *Let*

$$t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \textit{so that } \{t\} = \begin{array}{|cc|}\hline \mathit{1} & \mathit{2} \\ \hline \mathit{3} & \mathit{4} \\ \hline \end{array} .$$

Then

$$V_t = \{(1), (24), (13), (13)(24)\}$$

and so

$$\mathcal{V}_t = (1) - (24) - (13) + (13)(24).$$

Applying $\mathcal{V}_t$ to the tabloid $\{t\}$, we have

$$e_t = \mathcal{V}_t.\{t\} = \begin{array}{|cc|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} - \begin{array}{|cc|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline\end{array} - \begin{array}{|cc|}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline\end{array} + \begin{array}{|cc|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}.$$

Now let t be a standard Young tableau with shape λ, let $t_{[1]}, t_{[2]}, ..., t_{[k]}$ be the columns of t, and write $t = t_{[1]}t_{[2]}...t_{[k]}$. Let $t_{1,j}, t_{2,j}, ..., t_{\ell,j}$ be the entries of the the jth column of t and let $\Delta(t_{[j]})=\Delta\left(x_{t_{1,j}}, x_{t_{2,j}}, ..., x_{t_{\ell,j}}\right)$ be the *Vandermonde determinant* in the variables subscripted by the entries from the jth column of t.

Example 7. *Using the tableau t from Example 1,*

$$t_{[1]}t_{[2]} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 6 & 8 \\ \hline 9 & \\ \cline{1-1}\end{array}$$

and

$$\Delta(t_{[1]}) = \Delta\left(x_1, x_2, x_6, x_9\right) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_6 & x_9 \\ {x_1}^2 & {x_2}^2 & {x_6}^2 & {x_9}^2 \\ {x_1}^3 & {x_2}^3 & {x_6}^3 & {x_9}^3 \end{vmatrix}.$$

Finally, let $\Delta(t)$ be the polynomial $\Delta(t) = \Delta(t_{[1]})\Delta(t_{[2]})...\Delta(t_{[k]})$. In [Sp] it is shown that

$$\{\Delta(t) : t \text{ is a standard } \lambda\text{-tableau}\}$$

is a basis for an irreducible representation of S_n in $\mathcal{P}$ that is equivalent to the representation S^λ in $\mathbb{C}[\mathcal{T}^\lambda]$. The polynomials $\Delta(t)$ are called *Specht polynomials.*

5 Theorem

Let $\phi : \mathcal{T}^\lambda \to \mathcal{P}$ be the map that takes each tabloid $\{t\}$ to the monomial $m_t = \Pi_{\alpha=1}^{n} {x_\alpha}^{\theta(\alpha)}$ where $\theta(\alpha) = i - 1$ if α is in the ith row of $\{t\}$. Note that the domain of ϕ depends on λ; this dependence will not be reflected in our notation for the sake of clarity.

Example 8.

$$\{t\} = \begin{array}{|cc|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \qquad \phi(\{t\}) = {x_1}^0 {x_2}^0 {x_3}^1 {x_4}^1 = m_t.$$

Note that since tabloids are row-equivalent, $\phi(\{t\})$ is independent of the representative t of $\{t\}$ and so the map ϕ is well-defined. We then extend ϕ to $\mathbb{C}[\mathcal{T}^\lambda]$ by linearity.

Theorem 9.

$$\phi(e_t) = \Delta(t).$$

Remark 10. *There seems to no recorded proof of this result, certainly not in any of the standard references on the representation theory of the symmetric group such as [K-J], [S] or [F]. It appears (in a slightly different form) as Equation (1) in [P] followed by "as can easily be verified."*

6 Proof

We start by proving a series of lemmas and then combine them for the proof.

Lemma 11. $\phi{:}\mathbb{C}[\mathcal{T}^\lambda] \to \mathcal{P}$ *intertwines the representations of* S_n *on* $\mathbb{C}[\mathcal{T}^\lambda]$ *and* $\mathcal{P}$*. That is, if* $\tau \in S_n$*, then* $\phi(\tau.\{t\}){=}\tau.(\phi(\{t\}))$.

Proof. It is sufficient to consider the case that τ is a two-cycle since these generate S_n. If τ transposes two entries of $\{t\}$, it also transposes the corresponding two variables in the monomial m_t. So, by (1), $\phi(\tau.\{t\}) = \tau.m_t = \tau.\phi(\{t\})$. □

Lemma 12. $\phi(t_{[1]}t_{[2]}\cdots t_{[k]}){=}\phi(t_{[1]})\phi(t_{[2]})\cdots\phi(t_{[k]})$.

Proof. This is just a reflection of the fact that multiplication of the variables is commutative and associative. In order to illustrate the main ideas and avoid drowning in notation, we prove this for the case for $t = t_{[1]}t_{[2]}$. It will be convenient to slightly abuse notation and write $\alpha \in t$ if the number α is an entry in a tableau t.

$$\phi(t_{[1]}t_{[2]}) = \prod_{\alpha\in t_{[1]}t_{[2]}} x_\alpha^{\theta(\alpha)} = \prod_{r\in t_{[1]}, s\in t_{[2]}} x_r^{\theta(r)} x_s^{\theta(s)}$$
$$= \prod_{r\in t_{[1]}} x_r^{\theta(r)} \prod_{s\in t_{[2]}} x_s^{\theta(s)} = \phi(t_{[1]})\phi(t_{[2]}).$$

The general result follows by induction on the columns of t. □

Lemma 13. $V_t = V_{t_{[1]}} \times V_{t_{[2]}} \times \cdots \times V_{t_{[k]}}$ *as a direct product of subgroups of* S_n.

Proof. Clearly the direct product $V_{t_{[1]}} \times V_{t_{[2]}} \times \ldots \times V_{t_{[k]}}$ is contained in V_t, since each factor $V_{t_{[i]}}$ preserves a column.

Now suppose $\sigma \in V_t$ and write $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ as a product of two-cycles. Since any entry in t can occur only once and since σ preserves only the columns of t, any two-cycles that preserve distinct columns must be disjoint. Since disjoint cycles commute, we can reorder and regroup the two-cycles in the above product σ so that each grouping preserves a distinct column.

Finally, since $V_{t_{[r]}} \bigcap V_{t_{[s]}} = \{(1)\}$ for $r \neq s$, this product is direct. Thus, V_t is contained in the direct product. □

Lemma 14. $\mathcal{V}_t = \mathcal{V}_{t_{[1]}} \cdot \mathcal{V}_{t_{[2]}} \cdot \ \cdots \ \cdot \mathcal{V}_{t_{[k]}}$*, the product on the right being taken in* $\mathbb{C}[S_n]$.

Proof.

$\mathcal{V}_{t_{[1]}} \times \mathcal{V}_{t_{[2]}}$	$=$	$\sum_{\sigma_1 \in V_{t_{[1]}}} \operatorname{sgn}(\sigma_1)\, \sigma_1 \times \sum_{\sigma_2 \in V_{t_{[2]}}} \operatorname{sgn}(\sigma_2)\, \sigma_2$	by definition,
	$=$	$\sum_{\sigma_1 \in V_{t_{[1]}}, \sigma_2 \in V_{t_{[2]}}} \operatorname{sgn}(\sigma_1) \operatorname{sgn}(\sigma_2)\, \sigma_1 \sigma_2$	by the distributive property,
	$=$	$\sum_{\sigma_1 \in V_{t_{[1]}}, \sigma_2 \in V_{t_{[2]}}} \operatorname{sgn}(\sigma_1 \sigma_2)\, \sigma_1 \sigma_2$	a property of sgn,
	$=$	$\sum_{\sigma_1 \sigma_2 \in V_{t_{[1]}} \times V_{t_{[2]}}} \operatorname{sgn}(\sigma_1 \sigma_2)\, \sigma_1 \sigma_2$	by definition of direct product,
	$=$	$\sum_{\sigma \in V_{t_{[1]} t_{[2]}}} \operatorname{sgn}(\sigma)\, \sigma$	by Lemma 13 with $\sigma = \sigma_1 \sigma_2$,
	$=$	$\mathcal{V}_{t_{[1]} t_{[2]}}$	by definition.

The general result follows by induction. □

Lemma 15. $\mathcal{V}_{t_{[r]}}.(t_{[s]}) = t_{[s]}$ *if* $r \neq s$

Proof. by definition, $V_{t_{[r]}}$ permutes only the entries in the rth column. Equivalently, $\mathcal{V}_{t_{[r]}}(m_{t_{[s]}}) = m_{t_{[s]}}$ if $r \neq s$. □

(Proof of Theorem 9). We begin by recalling Leibniz' formula for determinant of an $n \times n$ matrix $(a_{i,j})$,

$$\operatorname{Det}(a_{i,j}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \Pi_{i=1}^{n} a_{i,\sigma(i)}.$$

In this language, the Vandermonde determinant becomes

$$\Delta(x_1, \dots, x_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \Pi_{i=1}^{n} x_{\sigma(i)}^{i-1}.$$

Applying this to the the jth column of a tableau t we have

$$\Delta(t_{[j]}) = \sum_{\sigma \in V_{t_{[j]}}} \operatorname{sgn}(\sigma) \prod_{i=1}^{\ell_j} x_{\sigma(t_{i,j})}{}^{i-1}. \tag{1}$$

where the upper index ℓ_j is the length of $t_{[j]}$.
We also note that

$$\begin{aligned} \phi(t_{[j]}) &= \textstyle\prod_{\alpha \in t_{[j]}} x_\alpha^{\theta(\alpha)} && \text{where } \theta(\alpha) = i-1 \text{ if } \alpha \text{ is in the } i\text{th row of } t_{[j]} \\ &= \textstyle\prod_{i=1}^{\ell_j} x_{t_{i,j}}^{i-1} && \text{since } t_{i,j} \text{ is in the } i\text{th row of } t_{[j]}. \end{aligned} \tag{2}$$

And thus

$$\begin{aligned}
\phi\left(e_{t_{[j]}}\right) &= \phi\left(\mathcal{V}_{t_{[j]}}.t_{[j]}\right) && \text{by definition of } e_{t_{[j]}}, \\
&= \phi\left(\textstyle\sum_{\sigma\in V_{t_{[j]}}} \operatorname{sgn}(\sigma)\,\sigma.t_{[j]}\right) && \text{by definition of } \mathcal{V}_{t_{[j]}}, \\
&= \textstyle\sum_{\sigma\in V_{t_{[j]}}} \operatorname{sgn}(\sigma)\,\sigma.\phi\left(t_{[j]}\right) && \text{by Lemma 11,} \\
&= \textstyle\sum_{\sigma\in V_{t_{[j]}}} \operatorname{sgn}(\sigma)\ \prod_{i=1}^{\ell_j} {x_{\sigma(t_{i,j})}}^{i-1} && \text{by Equation 2,} \\
&= \Delta(t_{[j]}) && \text{by Equation 1.}
\end{aligned} \tag{3}$$

Finally we combine these results:

$$\begin{aligned}
\phi(e_t) &= \phi(\mathcal{V}_t.\{t\}) && \text{by definition of } e_t, \\
&= \mathcal{V}_t.\phi(\{t\}) && \text{by Lemma 11,} \\
&= [\mathcal{V}_{t_{[1]}}\times\cdots\times\mathcal{V}_{t_{[k]}}].\,\phi(t_{[1]}\cdots t_{[k]}) && \text{by Lemma 11,} \\
&= [\mathcal{V}_{t_{[1]}}\times\cdots\times\mathcal{V}_{t_{[k]}}].\,\phi(t_{[1]})\cdots\phi(t_{[k]}) && \text{by Lemma 12,} \\
&= \mathcal{V}_{t_{[1]}}.\,\phi(t_{[1]})\cdots\mathcal{V}_{t_{[k]}}.\,\phi(t_{[k]}) && \text{by Lemma 15,} \\
&= \phi(\mathcal{V}_{t_{[1]}}.\,t_{[1]})\cdots\phi(\mathcal{V}_{t_{[k]}}.\,t_{[k]}) && \text{by Lemma 11,} \\
&= \phi(e_{t_{[1]}})\cdots\phi(e_{t_{[k]}}) && \text{by definition of } e_{t_{[j]}}, \\
&= \Delta(t_{[1]})\cdots\Delta(t_{[k]}) && \text{by Equation 3,} \\
&= \Delta(t) && \text{by definition of } \Delta(t),
\end{aligned}$$

as desired. □

7 Example

Recall t, $\{t\}$, V_t, and $\mathcal{V}_t$ from Example 6, and recall that

$$e_t = \mathcal{V}_t(\{t\}) = \begin{array}{|cc|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} - \begin{array}{|cc|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline\end{array} - \begin{array}{|cc|}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline\end{array} + \begin{array}{|cc|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}.$$

Therefore

$$\begin{aligned}
\phi(e_t) &= {x_1}^0{x_2}^0{x_3}^1{x_4}^1 - {x_2}^0{x_3}^0{x_1}^1{x_4}^1 - {x_1}^0{x_4}^0{x_2}^1{x_3}^1 + {x_3}^0{x_4}^0{x_1}^1{x_2}^1 \\
&= x_1x_2 + x_3x_4 - x_2x_3 - x_1x_4.
\end{aligned}$$

And by definition of the Specht polynomial,

$$\begin{aligned}
\Delta(t) &= \begin{vmatrix} 1 & 1 \\ x_1 & x_3 \end{vmatrix} \times \begin{vmatrix} 1 & 1 \\ x_2 & x_4 \end{vmatrix} \\
&= (x_3 - x_1)\times(x_4 - x_2) \\
&= x_1x_2 + x_3x_4 - x_2x_3 - x_1x_4.
\end{aligned}$$

Remark 16. *This result generalizes. We can assign an exponent to each row of a tabloid, and then raise the variables sub-scripted by each entry in that row to the assigned exponent. For example, if we place the assigned exponents in the leftmost column, then the tabloid*

1	1	3	5	7
4	2	4		
3	6	8		
7	9			

maps to the monomial $x_1 x_3 x_5 x_7 x_2^4 x_4^4 x_6^3 x_8^3 x_9^7$.

In this way we can obtain basis vectors for irreducible representations in polynomial spaces of any degree.

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