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Another view of the coarse invariant σ

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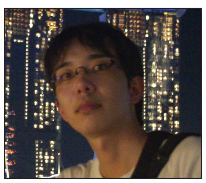

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Abstract Miller, Stibich and Moore [6] developed a set-valued coarse invariant $\sigma(X,\xi)$ of pointed metric spaces. DeLyser, LaBuz and Tobash [2] provided a different way to construct $\sigma(X,\xi)$ (as the set of all sequential ends). This paper provides yet another definition of $\sigma(X,\xi)$. To do this, we introduce a metric on the set $S(X,\xi)$ of coarse maps $(\mathbb{N},0) \to (X,\xi)$, and prove that $\sigma(X,\xi)$ is equal to the set of coarsely connected components of $S(X,\xi)$. As a by-product, our reformulation trivialises some known theorems on $\sigma(X,\xi)$, including the functoriality and the coarse invariance.

1 Introduction

Miller, Stibich and Moore [6] developed a set-valued coarse invariant $\sigma(X,\xi)$ of σ-stable pointed metric spaces (X,ξ). DeLyser, LaBuz and Wetsell [3] generalised it to pointed metric spaces (without σ-stability). The coarse invariance of $\sigma(X,\xi)$ was proved by Fox, LaBuz and Laskowsky [4] for σ-stable spaces, and by DeLyser, LaBuz and Wetsell [3] for general spaces.

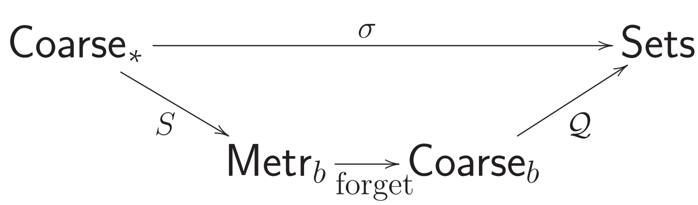
We start with recalling the definition of $\sigma(X,\xi)$. We adopt a simplified definition given by DeLyser, LaBuz and Tobash [2]. Let (X,ξ) be a pointed metric space. A *coarse sequence* in (X,ξ) is a coarse map $s : (\mathbb{N},0) \to (X,\xi)$. Denote the set of coarse sequences in (X,ξ) by $S(X,\xi)$. Given $s,t \in S(X,\xi)$, we write $s \sqsubseteq^{\sigma}_{X,\xi} t$ if s is a subsequence of t. Denote the equivalence closure of $\sqsubseteq^{\sigma}_{X,\xi}$ by $\equiv^{\sigma}_{X,\xi}$. In other words, $s \equiv^{\sigma}_{X,\xi} t$ if and only if there exists a finite sequence $\{u_i\}_{i=0}^{n}$ in $S(X,\xi)$ such that $u_0 = s$, $u_n = t$, and $u_i \sqsubseteq^{\sigma}_{X,\xi} u_{i+1}$ or $u_{i+1} \sqsubseteq^{\sigma}_{X,\xi} u_i$ for all $i < n$. The desired invariant is then defined as the quotient set:

$$\begin{aligned}\sigma(X,\xi) &:= S(X,\xi)/\equiv^{\sigma}_{X,\xi}\\ &:= \{[s]^{\sigma}_{X,\xi} \,|\, s \in S(X,\xi)\},\end{aligned}$$

where $[s]^{\sigma}_{X,\xi}$ is the $\equiv^{\sigma}_{X,\xi}$-equivalence class of s. As noted in [5], there is no difficulty in generalising $\sigma(X,\xi)$ to pointed coarse spaces (X,ξ). See the subsection **Notation and terminology** below for the definitions of the terms used here.

DeLyser, LaBuz and Tobash [2] provided an alternative definition of $\sigma(X,\xi)$. Suppose (X,ξ) is a pointed metric space. Two coarse sequences $s,t \in S(X,\xi)$ are said to *converge to the same sequential end* (and denoted by $s \equiv^{e}_{X,\xi} t$) if there is a $K > 0$ such that for all bounded subsets B of X there is an $N \in \mathbb{N}$ such that $\{s(i) \,|\, i \geq N\}$ and $\{t(i) \,|\, i \geq N\}$ are contained in the same K-chain-connected component of $X \setminus B$. The $\equiv^{e}_{X,\xi}$-equivalence classes are called *sequential ends* in (X,ξ). It was proved that $\equiv^{\sigma}_{X,\xi}$ and $\equiv^{e}_{X,\xi}$ coincides. As a result, $\sigma(X,\xi)$ is equal to the set of sequential ends in (X,ξ). This gives another view of $\sigma(X,\xi)$.

This paper aims to provide yet another view of $\sigma(X,\xi)$. Consider the following diagram:



where Coarse_* is the category of pointed coarse spaces and (base point preserving) coarse maps, Metr_b the category of metric spaces and bornologous maps, Coarse_b the category of coarse spaces and bornologous maps, and Sets the category of sets and maps. In Section 2, we introduce the so-called coarsely connected component functor $\mathcal{Q} : \mathsf{Coarse}_b \to \mathsf{Sets}$. The coarse invariance of $\mathcal{Q}$ is proved. In Section 3, we introduce a metric on the set $S(X,\xi)$, where the metric is allowed to take the value ∞. This forms a functor $S : \mathsf{Coarse}_* \to \mathsf{Metr}_b$. We prove the preservation of bornotopy by S. In Section 4, we prove that σ can be considered as the composition of the two functors $\mathcal{Q}$ and S, which commutes the above diagram. As a by-product, our reformulation trivialises some known theorems on $\sigma(X,\xi)$, including the functoriality and the coarse invariance.

Notation and terminology

Let $f, g: X \to Y$ be maps, E, F binary relations on X (i.e. subsets of $X \times X$), and $n \in \mathbb{N}$. Then

$$\begin{aligned} E \circ F &:= \{(x,y) \in X \times X \mid (x,z) \in E \text{ and } (z,y) \in F \text{ for some } z \in X\}, \\ E^{-1} &:= \{(y,x) \in X \times X \mid (x,y) \in E\}, \\ E^0 &:= \Delta_X := \{(x,x) \mid x \in X\}, \\ E^{n+1} &:= E^n \circ E, \\ (f \times g)(E) &:= \{(f(x), g(y)) \mid (x,y) \in E\}. \end{aligned}$$

A *coarse structure* on a set X is a family $\mathcal{C}_X$ of binary relations on X with the following properties:

1. $\Delta_X \in \mathcal{C}_X$;
2. $E \subseteq F \in \mathcal{C}_X \implies E \in \mathcal{C}_X$; and
3. $E, F \in \mathcal{C}_X \implies E \cup F, E \circ F, E^{-1} \in \mathcal{C}_X$.

A set equipped with a coarse structure is called a *coarse space*. A subset A of X is called a *bounded set* if $A \times A \in \mathcal{C}_X$. We denote the family of bounded subsets of X by $\mathcal{B}_X$. This family satisfies the following:

1. $\bigcup \mathcal{B}_X = X$;
2. $A \subseteq B \in \mathcal{B}_X \implies A \in \mathcal{B}_X$;
3. $A, B \in \mathcal{B}_X, A \cap B \neq \varnothing \implies A \cup B \in \mathcal{B}_X$.

A typical example of a coarse structure is the *bounded coarse structure* induced by a metric $d_X : X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$:

$$\mathcal{C}_{d_X} := \{E \subseteq X \times X \mid \sup d_X(E) < \infty\} \cup \{\varnothing\}.$$

Then the boundedness defined above agrees with the usual boundedness. We assume that every metric space is endowed with the bounded coarse structure throughout this paper.

Let $f, g : X \to Y$ be maps from a set X to a coarse space Y. We say that f and g are *bornotopic* (or *close*) if $(f \times g)(\Delta_X) \in \mathcal{C}_Y$. Obviously bornotopy gives an equivalence relation on the set Y^X of all maps from X to Y.

Suppose $f : X \to Y$ is a map between coarse spaces X, Y. Then f is said to be

1. *proper* if $f^{-1}(B) \in \mathcal{B}_X$ for all $B \in \mathcal{B}_Y$;
2. *bornologous* if $(f \times f)(E) \in \mathcal{C}_Y$ for all $E \in \mathcal{C}_X$;
3. *coarse* if it is proper and bornologous;
4. an *asymorphism* (or an *isomorphism of coarse spaces*) if it is a bornologous bijection such that the inverse map is also bornologous;

5. a *coarse equivalence* (or a *bornotopy equivalence*) if it is bornologous, and there exists a bornologous map $g : Y \to X$ (called a *coarse inverse* or a *bornotopy inverse* of f) such that $g \circ f$ and $f \circ g$ are bornotopic to the identity maps id_X and id_Y, respectively.

For more information, see the monograph [7] by John Roe.

2 Coarsely connected components

Let X be a coarse space. A subset A of X is said to be *coarsely connected* if $\{x, y\} \in \mathcal{B}_X$ for all $x, y \in A$ ([7, Definition 2.11]). For $x \in X$, we set

$$\mathcal{Q}_X(x) := \bigcup_{x \in B \in \mathcal{B}_X} B,$$

and call it the *coarsely connected component* of X containing x. It is easy to see that $\mathcal{Q}_X(x)$ is the largest coarsely connected subset of X that contains x (see also [7, Remark 2.20]). We denote the set of all coarsely connected components of X by $\mathcal{Q}(X)$:

$$\mathcal{Q}(X) := \{\mathcal{Q}_X(x) \,|\, x \in X\}.$$

Lemma 1. *Let $f : X \to Y$ be a bornologous map. If X is coarsely connected, then so is the image $f(X)$.*

Proof. The statement is immediate from the fact that every bornologous map preserves boundedness. □

Theorem 2 (Functoriality)**.** *Every bornologous map $f : X \to Y$ functorially induces a map $\mathcal{Q}(f) : \mathcal{Q}(X) \to \mathcal{Q}(Y)$ by $\mathcal{Q}(f)(\mathcal{Q}_X(x)) := \mathcal{Q}_Y(f(x))$.*

Proof. It suffices to verify the well-definedness. Let $x, y \in X$ and suppose $\mathcal{Q}_X(x) = \mathcal{Q}_X(y)$. Since f is bornologous and $\mathcal{Q}_X(x)$ is coarsely connected, $f(\mathcal{Q}_X(x))$ is coarsely connected and contains $f(x)$. By the maximality of $\mathcal{Q}_Y(f(x))$, we have that $f(y) \in f(\mathcal{Q}_X(y)) = f(\mathcal{Q}_X(x)) \subseteq \mathcal{Q}_Y(f(x))$. By the maximality of $\mathcal{Q}_Y(f(y))$, we have that $\mathcal{Q}_Y(f(x)) \subseteq \mathcal{Q}_Y(f(y))$. By symmetry, $\mathcal{Q}_Y(f(y)) \subseteq \mathcal{Q}_Y(f(x))$ holds. Therefore $\mathcal{Q}_Y(f(x)) = \mathcal{Q}_Y(f(y))$. □

Theorem 3 (Coarse invariance)**.** *If two bornologous maps $f, g : X \to Y$ are bornotopic, then $\mathcal{Q}(f) = \mathcal{Q}(g)$.*

Proof. The proof is similar to that of Theorem 2. Let $x \in X$. Since f and g are bornotopic, $(f(x), g(x)) \in (f \times g)(\Delta_X) \in \mathcal{C}_Y$, so $\{f(x), g(x)\}$ is bounded in Y. Thus $\{f(x), g(x)\}$ is coarsely connected and contains $f(x)$. By the maximality of $\mathcal{Q}_Y(f(x))$, we have that $g(x) \in \{f(x), g(x)\} \subseteq \mathcal{Q}_Y(f(x))$. By the maximality of $\mathcal{Q}_Y(g(x))$, we have that $\mathcal{Q}_Y(f(x)) \subseteq \mathcal{Q}_Y(g(x))$. The reverse inclusion $\mathcal{Q}_Y(g(x)) \subseteq \mathcal{Q}_Y(f(x))$ holds by symmetry. It follows that $\mathcal{Q}(f)(\mathcal{Q}_X(x)) = \mathcal{Q}_Y(f(x)) = \mathcal{Q}_Y(g(x)) = \mathcal{Q}(g)(\mathcal{Q}_X(x))$. □

3 Metrisation of $S(X,\xi)$

Let (X,ξ) be a pointed coarse space. A coarse map $s:(\mathbb{N},0)\to(X,\xi)$ is called a *coarse sequence* in (X,ξ). Denote by $S(X,\xi)$ the set of all coarse sequences of (X,ξ). In the preceding studies [6, 4, 3, 2], $S(X,\xi)$ is just a set with no structure. In fact, as we shall see below, $S(X,\xi)$ has a geometric structure relevant to $\sigma(X,\xi)$. We define a metric $d_{S(X,\xi)}:S(X,\xi)\times S(X,\xi)\to\mathbb{N}\cup\{\infty\}$ on $S(X,\xi)$ as follows:

$$d_{S(X,\xi)}(s,t):=\inf\{n\in\mathbb{N}|\,(s,t)\in\left(\sqsubseteq^{\sigma}_{X,\xi}\cup\sqsupseteq^{\sigma}_{X,\xi}\right)^{n}\},$$

where $\inf\varnothing:=\infty$. It is easy to check that $d_{S(X,\xi)}$ is a metric. Thus $S(X,\xi)$ is equipped with a coarse structure, viz., the bounded coarse structure induced by $d_{S(X,\xi)}$.

Lemma 4. *Let (X,ξ) be a pointed coarse space and $s,t\in S(X,\xi)$.*

1. *The following are equivalent:*
   - (a) *$s\equiv^{\sigma}_{X,\xi}t$;*
   - (b) *$d_{S(X,\xi)}(s,t)\in\mathbb{N}$;*
   - (c) *there exists a sequence $\{u_i\}_{i=0}^{n}$ in $S(X,\xi)$ of length $n+1$ such that $u_0=s$, $u_n=t$ and $d_{S(X,\xi)}(u_i,u_{i+1})=1$ for all $i<n$, where n is an arbitrary constant greater than or equal to $d_{S(X,\xi)}(s,t)$.*
2. *The following are equivalent:*
   - (a) *$s\not\equiv^{\sigma}_{X,\xi}t$;*
   - (b) *$d_{S(X,\xi)}(s,t)=\infty$;*
   - (c) *there is no finite sequence $\{u_i\}_{i=0}^{n}$ in $S(X,\xi)$ such that $u_0=s$, $u_n=t$ and $d_{S(X,\xi)}(u_i,u_{i+1})=1$ for all $i<n$.*

Proof. Notice that $d_{S(X,\xi)}(s,t)\le n$ if and only if there exists a sequence $\{u_i\}_{i=0}^{n}$ in $S(X,\xi)$ of length $n+1$ such that $u_0=s$, $u_n=t$, and $u_i\sqsubseteq^{\sigma}_{X,\xi}u_{i+1}$ or $u_{i+1}\sqsubseteq^{\sigma}_{X,\xi}u_i$ for all $i<n$. Also, note that $d_{S(X,\xi)}(s,t)=\infty$ if and only if there is no such finite sequence in $S(X,\xi)$. The above equivalences are now obvious. □

Theorem 5 (Functoriality)**.** *Each coarse map $f:(X,\xi)\to(Y,\eta)$ functorially induces a bornologous map $S(f):S(X,\xi)\to S(Y,\eta)$ by $S(f)(s):=f\circ s$.*

Proof. Well-definedness: let $s\in S(X,\xi)$. Clearly $S(f)(s)$ is a map from $(\mathbb{N},0)$ to (Y,η). The class of coarse maps is closed under composition, so $S(f)(s)$ is coarse. (Let $E\in\mathcal{C}_{\mathbb{N}}$. Then $(s\times s)(E)\in\mathcal{C}_X$ by the bornologousness of s, so $(f\circ s\times f\circ s)(E)=(f\times f)((s\times s)(E))\in\mathcal{C}_Y$ by the bornologousness of f. Let $B\in\mathcal{B}_Y$. Then $f^{-1}(B)\in\mathcal{B}_X$ by the properness of f, and hence $(f\circ s)^{-1}(B)=s^{-1}\circ f^{-1}(B)\in\mathcal{B}_{\mathbb{N}}$ by the properness of s.) Hence $S(f)(s)\in S(Y,\eta)$.

Bornologousness: Let $s,t\in S(X,\xi)$ and suppose $d_{S(X,\xi)}(s,t)\le n$, i.e., there is a sequence $\{u_i\}_{i=0}^{n}$ in $S(X,\xi)$ of length $n+1$ such that $u_0=s$, $u_n=t$, and $u_i\sqsubseteq^{\sigma}_{X,\xi}u_{i+1}$ or $u_{i+1}\sqsubseteq^{\sigma}_{X,\xi}u_i$ for all $i<n$. Then the sequence $\{f\circ u_i\}_{i=0}^{n}$ witnesses that $d_{S(Y,\eta)}(S(f)(s),S(f)(t))=d_{S(Y,\eta)}(f\circ s,f\circ t)\le n$. □

Theorem 6 (Preservation of bornotopy)**.** *If coarse maps $f, g : (X, \xi) \to (Y, \eta)$ are bornotopic, then so are $S(f), S(g) : S(X, \xi) \to S(Y, \eta)$.*

Proof. Let $s \in S(X, \xi)$. We define a map $t : (\mathbb{N}, 0) \to (Y, \eta)$ as follows:

$$t(i) := \begin{cases} S(f)(s)(j), & i = 2j, \\ S(g)(s)(j), & i = 2j + 1. \end{cases}$$

Let us verify that $t \in S(Y, \eta)$. Firstly, let $B \in \mathcal{B}_Y$. Then

$$t^{-1}(B) = 2(S(f)(s))^{-1}(B) \cup \left(2(S(g)(s))^{-1}(B) + 1\right).$$

Since $S(f)(s)$ and $S(g)(s)$ are proper, the two sets $2(S(f)(s))^{-1}(B)$ and $2(S(g)(s))^{-1}(B) + 1$ are bounded in $\mathbb{N}$ (i.e. finite), so $t^{-1}(B) \in \mathcal{B}_{\mathbb{N}}$. Therefore t is proper. Secondly, let $n \in \mathbb{N}$. Since $S(f)(s)$ and $S(g)(s)$ are bornologous, there exists an $E \in \mathcal{C}_Y$ such that $(S(f)(s)(i), S(f)(s)(j)) \in E$ and $(S(g)(s)(i), S(g)(s)(j)) \in E$ hold for all $i, j \in \mathbb{N}$ with $|i - j| \le n$. Since f and g are bornotopic,

$$\begin{aligned} F &:= \{(S(f)(s)(i), S(g)(s)(i)) \,|\, i \in \mathbb{N}\} \\ &= \{(f \circ s(i), g \circ s(i)) \,|\, i \in \mathbb{N}\} \\ &\subseteq (f \times g)(\Delta_X) \\ &\in \mathcal{C}_Y. \end{aligned}$$

Then $(S(f)(s)(i), S(g)(s)(j)) \in E \circ F \in \mathcal{C}_Y$ and $(S(g)(s)(i), S(f)(s)(j)) \in E \circ F^{-1} \in \mathcal{C}_Y$ hold for all $i, j \in \mathbb{N}$ with $|i - j| \le n$. Now let $G := E \cup (E \circ F) \cup \left(E \circ F^{-1}\right) \in \mathcal{C}_Y$. Then $(t(i), t(j)) \in G$ holds for all $i, j \in \mathbb{N}$ with $|i - j| \le n$. Therefore t is bornologous.

Both $S(f)(s)$ and $S(g)(s)$ are subsequences of t, i.e., $S(f)(s) \sqsubseteq^{\sigma}_{Y,\eta} t \sqsupseteq^{\sigma}_{Y,\eta} S(g)(s)$, so $d_{S(Y,\eta)}(S(f)(s), S(g)(s)) \le 2$. Hence

$$\begin{aligned} (S(f) \times S(g))\left(\Delta_{S(X,\xi)}\right) &\subseteq \{(u, v) \in S(Y, \eta) \times S(Y, \eta) \,|\, d_{S(Y,\eta)}(u, v) \le 2\} \\ &\in \mathcal{C}_{S(Y,\eta)}. \end{aligned}$$

Therefore $S(f)$ and $S(g)$ are bornotopic. □

The next theorem shows that the base point can be replaced with any other point lying in the same coarsely connected component.

Theorem 7 (Changing the base point)**.** *Let X be a coarse space, and $\xi_1, \xi_2 \in X$. If $\mathcal{Q}_X(\xi_1) = \mathcal{Q}_X(\xi_2)$, then $S(X, \xi_1)$ and $S(X, \xi_2)$ are isometric.*

Proof. Define maps $T_{21} : S(X, \xi_1) \to S(X, \xi_2)$ and $T_{12} : S(X, \xi_2) \to S(X, \xi_1)$

by

$$T_{21}(s)(i) := \begin{cases} \xi_2, & i = 0, \\ s(i), & i > 0, \end{cases}$$

$$T_{12}(t)(i) := \begin{cases} \xi_1, & i = 0, \\ t(i), & i > 0. \end{cases}$$

We first verify the well-definedness, i.e. $T_{21}(s) \in S(X, \xi_2)$ and $T_{12}(t) \in S(X, \xi_1)$. Obviously $T_{21}(s)(0) = \xi_2$. Let $B \in \mathcal{B}_X$. Then $(T_{21}(s))^{-1}(B) \subseteq s^{-1}(B) \cup \{0\}$, where $s^{-1}(B)$ is bounded in $\mathbb{N}$ (i.e. finite) by the properness of s, so $(T_{21}(s))^{-1}(B)$ is also bounded in $\mathbb{N}$. Hence $T_{21}(s)$ is proper. Next, let $E \in \mathcal{C}_{\mathbb{N}}$. For each $(i, j) \in E$, there are the following possibilities:

Case 1. $i = j = 0$.
Then $T_{21}(s)(i) = \xi_2 = T_{21}(s)(j)$, so $(T_{21}(s)(i), T_{21}(s)(j)) \in \Delta_X \in \mathcal{C}_X$.

Case 2. $i = 0$ and $j \neq 0$.
In this case, $T_{21}(s)(i) = \xi_2$, $s(i) = \xi_1$ and $s(j) = T_{21}(s)(j)$.
So $(T_{21}(s)(i), T_{21}(s)(j)) \in \{(\xi_2, \xi_1)\} \circ (s \times s)(E) \in \mathcal{C}_X$.

Case 3. $i \neq 0$ and $j = 0$.
Similar to the above case, we have $(T_{21}(s)(i), T_{21}(s)(j)) \in (s \times s)(E) \circ \{(\xi_1, \xi_2)\} \in \mathcal{C}_X$.

Case 4. $i \neq 0$ and $j \neq 0$.
Then $T_{21}(s)(i) = s(i)$ and $T_{21}(s)(j) = s(j)$, whence we have
$(T_{21}(s)(i), T_{21}(s)(j)) \in (s \times s)(E) \in \mathcal{C}_X$.

Set $F := \Delta_X \cup (\{(\xi_2, \xi_1)\} \circ (s \times s)(E)) \cup ((s \times s)(E) \circ \{(\xi_1, \xi_2)\}) \cup (s \times s)(E)$. Then $(T_{21}(s), T_{21}(s))(E) \subseteq F \in \mathcal{C}_X$, so $(T_{21}(s), T_{21}(s))(E) \in \mathcal{C}_X$. Hence $T_{21}(s)$ is bornologous. Since the definitions are symmetric, the same argument applies to $T_{12}(t)$.

Clearly $T_{12} \circ T_{21} = \mathrm{id}_{S(X, \xi_1)}$ and $T_{21} \circ T_{12} = \mathrm{id}_{S(X, \xi_2)}$. It suffices to prove that T_{21} is an isometry.

Let $s, t \in S(X, \xi_1)$ and suppose that $s \sqsubseteq^{\sigma}_{X,\xi} t$, i.e., there is a strictly monotone function $\kappa : \mathbb{N} \to \mathbb{N}$ such that $s = t \circ \kappa$. Since $\kappa(i) \geq i$, we have $T_{21}(s)(i) = s(i) = t(\kappa(i)) = T_{21}(t)(\kappa(i))$ for all $i > 0$. Now, define

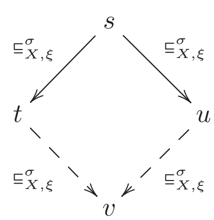
$$\kappa'(i) := \begin{cases} 0, & i = 0, \\ \kappa(i), & i > 0. \end{cases}$$

Then $T_{21}(s)(i) = T_{21}(t)(\kappa'(i))$ holds for all $i \in \mathbb{N}$ (including the case $i = 0$). Hence $T_{21}(s) \sqsubseteq^{\sigma}_{X,\xi_2} T_{21}(t)$. Note that, by symmetry, the same applies to T_{12}. Conversely, let $s, t \in S(X, \xi_1)$ and suppose $T_{21}(s) \sqsubseteq^{\sigma}_{X,\xi_2} T_{21}(t)$. Then $s = T_{12} \circ T_{21}(s) \sqsubseteq^{\sigma}_{X,\xi_1} T_{12} \circ T_{21}(t) = t$.

Now, let $s, t \in S(X, \xi_1)$ and suppose $d_{S(X,\xi_1)}(s, t) \leq n$, i.e., there is a sequence $\{u_i\}_{i=0}^{n}$ in $S(X,\xi)$ of length $n+1$ such that $u_0 = s$, $u_n = t$, and $u_i \sqsubseteq^{\sigma}_{X,\xi} u_{i+1}$ or $u_{i+1} \sqsubseteq^{\sigma}_{X,\xi} u_i$ for all $i < n$. By the previous paragraph, $T_{21}(u_i) \sqsubseteq^{\sigma}_{X,\xi} T_{21}(u_{i+1})$ or $T_{21}(u_{i+1}) \sqsubseteq^{\sigma}_{X,\xi} T_{21}(u_i)$ for all $i < n$. So $d_{S(X,\xi_2)}(T_{21}(s), T_{21}(t)) \leq n$. The same applies to T_{12} by symmetry. Conversely, let $s,t \in S(X,\xi_1)$ and suppose $d_{S(X,\xi_2)}(T_{21}(s), T_{21}(t)) \leq n$. Then it follows that $d_{S(X,\xi_1)}(s,t) = d_{S(X,\xi_1)}(T_{12} \circ T_{21}(s), T_{12} \circ T_{21}(t)) \leq n$. Consequently, both T_{21} and T_{12} are isometries. □

In fact, the metric function $d_{S(X,\xi)}$ only takes the values 0, 1, 2 and ∞. To show this fact, we need the "confluence" property of $\sqsubseteq^{\sigma}_{X,\xi}$.

Lemma 8 (DeLyser, LaBuz and Tobash [2, Lemma 3.1])**.** *Let $s,t,u \in S(X,\xi)$ and suppose $s \sqsubseteq^{\sigma}_{X,\xi} t,u$. Then there is a $v \in S(X,\xi)$ such that $t,u \sqsubseteq^{\sigma}_{X,\xi} v$.*



Proof. By the definition of "subsequence", there are strictly monotone functions $\kappa, \lambda : \mathbb{N} \to \mathbb{N}$ such that $s = t \circ \kappa = u \circ \lambda$. The desired sequence $v \in S(X,\xi)$ is given by

$$\underbrace{s(0), t(1), \ldots, t(\kappa(1)-1), s(1)}_{t(0),\ldots,t(\kappa(1))}, \underbrace{s(0), u(1), \ldots, u(\lambda(1)-1), s(1)}_{u(0),\ldots,u(\lambda(1))},$$

$$\underbrace{s(1), t(\kappa(1)+1), \ldots, t(\kappa(2)-1), s(2)}_{t(\kappa(1)),\ldots,t(\kappa(2))}, \underbrace{s(1), u(\lambda(1)+1), \ldots, u(\lambda(2)-1), s(2)}_{u(\lambda(1)),\ldots,u(\lambda(2))},$$

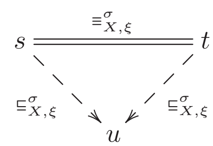
$$\vdots$$

Obviously v has t and u as subsequences. Let $E = \{(i,j) \mid |i-j| \leq 1\}$. (Note that $\mathcal{C}_{\mathbb{N}}$ is generated by $\{E^n | n \in \mathbb{N}\}$.) Since s, t and u are bornologous, $(s \times s)(E), (t \times t)(E), (u \times u)(E) \in \mathcal{C}_X$. Note that any two adjacent points $(v(i), v(i \pm 1))$ are one of the following forms:

$$(t(j), t(j \pm 1)),\ (s(j), s(j \pm 1)),\ (u(j), u(j \pm 1)),\ (s(j), s(j)),$$

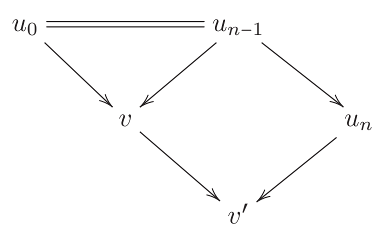
so $(v \times v)(E) \subseteq (t \times t)(E) \cup (s \times s)(E) \cup (u \times u)(E) \cup \Delta_X \in \mathcal{C}_X$. Hence v is bornologous. Similarly, one can easily prove that v is proper (i.e. diverges to infinity). □

Lemma 9 (DeLyser, LaBuz and Tobash [2, Proposition 3.2])**.** *Let* $s, t \in S(X, \xi)$ *and suppose* $s \equiv^\sigma_{X,\xi} t$*. Then there is a* $u \in S(X, \xi)$ *such that* $s, t \sqsubseteq^\sigma_{X,\xi} u$*.*

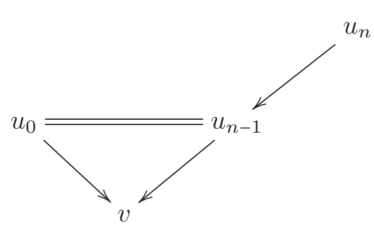


Proof. Choose a sequence $\{u_i\}_{i=0}^n$ in $S(X, \xi)$ such that $u_0 = s$, $u_n = t$, and $u_i \sqsubseteq^\sigma_{X,\xi} u_{i+1}$ or $u_{i+1} \sqsubseteq^\sigma_{X,\xi} u_i$ for all $i < n$. We show that there is a $v \in S(X, \xi)$ such that $u_0, u_n \sqsubseteq^\sigma_{X,\xi} v$ by induction on the length n. The base case $n = 0$ is trivial. Suppose $n > 0$. Since $u_0 \equiv^\sigma_{X,\xi} u_{n-1}$, there is a $v \in S(X, \xi)$ such that $u_0, u_{n-1} \sqsubseteq^\sigma_{X,\xi} v$ by the induction hypothesis.

Case 1. $u_{n-1} \sqsubseteq^\sigma_{X,\xi} u_n$.
Since $u_{n-1} \sqsubseteq^\sigma_{X,\xi} u_n, v$, there is a $v' \in S(X, \xi)$ such that $u_n, v \sqsubseteq^\sigma_{X,\xi} v'$ by Lemma 8. Then $u_0 \sqsubseteq^\sigma_{X,\xi} v \sqsubseteq^\sigma_{X,\xi} v'$, so $u_0 \sqsubseteq^\sigma_{X,\xi} v'$.



Case 2. $u_{n-1} \sqsupseteq^\sigma_{X,\xi} u_n$.
Then $u_n \sqsubseteq^\sigma_{X,\xi} u_{n-1} \sqsubseteq^\sigma_{X,\xi} v$, so $u_n \sqsubseteq^\sigma_{X,\xi} v$.



□

Theorem 10. $d_{S(X,\xi)} : X \times X \to \{0, 1, 2, \infty\}$.

Proof. Let $s, t \in S(X, \xi)$ and suppose $s \equiv^\sigma_{X,\xi} t$. There is a $u \in S(X, \xi)$ such that $s, t \sqsubseteq^\sigma_{X,\xi} u$ by Lemma 9. Hence $d_{S(X,\xi)}(s, t) \leq 2$. □

A similar argument in Lemma 9 is often used in the context of rewriting systems (such as lambda calculus). See also [1, Chapter 6].

4 Alternative definition of σ

Our main theorem is the following. This gives an alternative definition of $\sigma(X,\xi)$ in terms of the coarse structure of $S(X,\xi)$.

Theorem 11. *Let (X,ξ) be a pointed coarse space. Then $[s]^{\sigma}_{X,\xi} = \mathcal{Q}_{S(X,\xi)}(s)$ for all $s \in S(X,\xi)$. Hence $\sigma(X,\xi) = \mathcal{Q}(S(X,\xi))$.*

Proof. Let $s \in S(X,\xi)$. Then, by Lemma 4-(1), $[s]^{\sigma}_{X,\xi}$ is coarsely connected (in fact, 1-chain-connected) as a subset of $S(X,\xi)$, and contains s. Hence $[s]^{\sigma}_{X,\xi} \subseteq \mathcal{Q}_{S(X,\xi)}(s)$ by the maximality of $\mathcal{Q}_{S(X,\xi)}(s)$. Conversely, let $t \in \mathcal{Q}_{S(X,\xi)}(s)$. By Lemma 4-(2), $s \equiv^{\sigma}_{X,\xi} t$ must hold, and therefore $t \in [s]^{\sigma}_{X,\xi}$. Hence $\mathcal{Q}_{S(X,\xi)}(s) \subseteq [s]^{\sigma}_{X,\xi}$. □

This theorem yields quite simple and systematic proofs of some existing results on $\sigma(X,\xi)$.

Theorem 12. *Each coarse map $f : (X,\xi) \to (Y,\eta)$ functorially induces a map $\sigma(f) : \sigma(X,\xi) \to \sigma(Y,\eta)$ by $\sigma(f)\left([s]^{\sigma}_{X,\xi}\right) := [f \circ s]^{\sigma}_{Y,\eta}$.*

Proof. Immediate from Theorem 2, Theorem 5 and Theorem 11. □

Corollary 13 (Miller, Stibich and Moore [6, Theorem 10])**.** *If pointed coarse spaces (X,ξ) and (Y,η) are asymorphic, then $\sigma(X,\xi) \cong \sigma(Y,\eta)$.*

Proof. Obvious from the fact that every functor preserves isomorphisms. □

Theorem 14. *If coarse maps $f, g : (X,\xi) \to (Y,\eta)$ are bornotopic, then $\sigma(f) = \sigma(g)$.*

Proof. Immediate from Theorem 6, Theorem 3 and Theorem 11. □

Corollary 15 (DeLyser, LaBuz and Wetsell [3, Theorem 4])**.** *If pointed coarse spaces (X,ξ) and (Y,η) are coarsely equivalent, then $\sigma(X,\xi) \cong \sigma(Y,\eta)$.*

Proof. Let $f : (X,\xi) \to (Y,\eta)$ be a coarse equivalence with a coarse inverse $g : (Y,\eta) \to (X,\xi)$. Then $f \circ g$ and $g \circ f$ are bornotopic to $\mathrm{id}_{(Y,\eta)}$ and $\mathrm{id}_{(X,\xi)}$, respectively. By Theorem 12 and Theorem 14,

$$\begin{aligned}
\mathrm{id}_{\sigma(Y,\eta)} &= \sigma\left(\mathrm{id}_{(Y,\eta)}\right) \\
&= \sigma(f \circ g) \\
&= \sigma(f) \circ \sigma(g), \\
\mathrm{id}_{\sigma(X,\xi)} &= \sigma\left(\mathrm{id}_{(X,\xi)}\right) \\
&= \sigma(g \circ f) \\
&= \sigma(g) \circ \sigma(f),
\end{aligned}$$

so $\sigma(f)$ and $\sigma(g)$ are inverse to each other. Hence $\sigma(X,\xi) \cong \sigma(Y,\eta)$. □

Corollary 16 (DeLyser, LaBuz and Wetsell [3, Proposition 3])**.** *Let X be a coarse space, and $\xi_1, \xi_2 \in X$. If $\mathcal{Q}_X(\xi_1) = \mathcal{Q}_X(\xi_2)$, then $\sigma(X,\xi_1)$ and $\sigma(X,\xi_2)$ are equipotent (i.e. have the same cardinality).*

Proof. By Theorem 7, $S(X,\xi_1)$ and $S(X,\xi_2)$ are isometric and hence asymorphic. So $\sigma(X,\xi_1) = \mathcal{Q}(S(X,\xi_1)) \cong \mathcal{Q}(S(X,\xi_2)) = \sigma(X,\xi_2)$ by Theorem 2 and Theorem 11. □

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