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Exploring Bounds for the Frobenius Number

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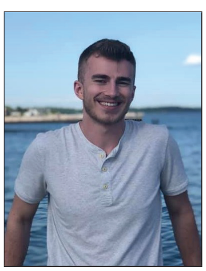

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Abstract Let G be a set of three natural numbers, $G = \{a, b, c\}$, such that $\gcd(a, b, c) = 1$. The Frobenius number of G is the largest integer that cannot be written as a non-negative linear combination of elements of G. In this article, we present some experimental results on the Frobenius number.

1 Introduction

Let $\mathbb{N}_0$ denote the set of non-negative integers. Given a set $G = \{a_1, a_2, \ldots, a_k\} \subset \mathbb{N}$ such that $\gcd(a_1, a_2, \ldots, a_k) = 1$, a natural number m is called representable if there exist non-negative integers $m_1, m_2, \ldots, m_k$ such that $m = \sum_{i=1}^{k} m_i a_i$. If no such coefficients can be found, m is called non-representable. Let $\mathrm{R}(G)$ be the set of all representable numbers and $\mathrm{NR}(G) = \mathbb{N}_0 \setminus \mathrm{R}(G)$ be the set of non-representable numbers. The linear Diophantine problem of Frobenius requires finding the largest integer in $\mathrm{NR}(G)$. This integer is called the Frobenius number and will be denoted by $f(G)$. It is well known and not hard to show that for all sufficiently large N, the equation $\sum_{i=1}^{k} m_i a_i = N$ has a solution with non-negative integers $m_1, m_2, \ldots, m_k$. Thus the Frobenius number $f(G)$ exists.

Although the origin of the problem is attributed to Sylvester, it seems Frobenius was instrumental in popularizing the problem in his lectures which associated the problem to his name. The problem of computing the Frobenius number dates back to the 19th century and it has a rich and long history. The interested reader can find a comprehensive survey covering different aspects of the problem in [1] and [5].

When $k = 2$, it is well known (most probably at least since Sylvester [6]) that $f(a_1, a_2) = a_1 a_2 - a_1 - a_2$. For $k > 2$, explicit formulas have not been established, except in special cases. However, there has been some work in the literature studying lower and upper bounds for the Frobenius number in some cases. In particular, when $G = \{x, y, z\}$, Davison [3] provided the lower bound

$$f(G) \geq L = \sqrt{3xyz} - x - y - z. \tag{1}$$

On the other hand, Beck et al. [2] conjectured the upper bound

$$f(G) \leq U = (\sqrt{xyz})^{\frac{5}{4}} - x - y - z. \tag{2}$$

For this upper bound, it was assumed that: i) one number in $\{x, y, z\}$ is not represented by the other two; ii) $x \nmid y + z$; and iii) $\{x, y, z\}$ is not of the form $\{a, ma+n, ma+2n\}$ for some $a, m, n \in \mathbb{N}$. The authors in [2] also conjectured that there exists an upper bound proportional to $(\sqrt{xyz})^p$, where $p < \frac{4}{3}$ in these cases.

In this paper, we further test the bounds given in (1) and (2) for certain types of sequences of three numbers. To simplify computations in [2], it was assumed that x, y, z are pairwise relatively prime but this assumption will not be used in the last two sequences presented in this paper.

Numerical semigroups

A numerical semigroup S is a subset of $\mathbb{N}_0$ that is closed under addition, contains 0, and has a finite complement in $\mathbb{N}_0$. It is known from [4] that $\mathrm{R}(G)$ is a numerical semigroup iff $\gcd(G) = 1$. We present a proof of this fact when G has two elements.

Lemma 1. *Let $G = \{a, b\}$. The set $\mathrm{NR}(G)$ is finite if and only if $\gcd(a, b) = 1$.*

Proof. Assume that neither a nor b is 1 since that would imply $\mathrm{NR}(G) = \emptyset$. Let $\gcd(a,b) = 1$. By Bezout's Lemma, there exist integers $x, y \in \mathbb{Z}$ such that $ax + by = 1$. Since $a \neq 1$ and $b \neq 1$ then x and y cannot be zero, which means one of them must be negative and the other must be positive. Without any loss of generality, assume y is negative and write $z = -y$. Thus, $ax = 1 - by = 1 + bz$. Now, letting $m = bz$ implies $ax = m + 1$. Note that bz and ax are $\mathbb{N}_0$-linear combinations of $\{a, b\}$ which implies $m, m+1 \in \mathrm{R}(G)$. The following is a sequence of m consecutive representable numbers:

$$m^2 = m \cdot m + 0 \cdot (m+1),\ m^2 + 1 = (m-1) \cdot m + 1 \cdot (m+1),$$
$$\ldots,\ m^2 + (m-1) = 1 \cdot m + (m-1)(m+1).$$

These numbers are linear combinations of $\{a, b\}$ since m, $m+1$ are. Now, any natural number $t > m^2 + (m-1)$ is also a linear combination of $\{a, b\}$ since t can be obtained by adding multiple(s) of m to one of the elements of the above sequence. The above shows that any natural number bigger than or equal to m^2 is a linear combination of $\{a, b\}$ which means that the non-representable numbers make a finite set.

Now, assume $\mathrm{NR}(G)$ is a finite set. Then there are two consecutive numbers $m, m+1$ that are representable. Write $m = ax + by$ and $m + 1 = az + bw$ for some x, y, z, $w \in \mathbb{N}_0$. Subtracting these two equations gives $1 = az + bw - ax - by = a(z-x) + b(w-y)$, and hence 1 is a $\mathbb{Z}$-linear combination of $\{a, b\}$, which implies that $\gcd(a, b) = 1$. □

If $S = \mathrm{R}(G)$ for some $G \subset \mathbb{N}$, we say S is generated by G and the Frobenius number of S is defined as before. Hence, it is the largest integer in $\mathbb{N}_0 \setminus S$ and will be denoted, from now on, by $f(S)$.

2 Numerical Semigroups from Arithmetic Sequences

Let $a \in \mathbb{N}$ and let S be the numerical semigroup generated by the arithmetic sequence $G = \{a, a+1, a+4\}$. The Frobenius number in this case can be found in [8, Theorem 1]:

$$f(S) = \begin{cases} \frac{1}{4}(a^2 + 8a - 4) & \text{if } a \equiv 0 \mod 4; \\ \frac{1}{4}(a^2 + 7a - 8) & \text{if } a \equiv 1 \mod 4; \\ \frac{1}{4}(a^2 + 6a - 12) & \text{if } a \equiv 2 \mod 4; \\ \frac{1}{4}(a^2 + 5a - 4) & \text{if } a \equiv 3 \mod 4. \end{cases} \tag{3}$$

This formula is used to test the lower and upper bounds given in equations (1) and (2). Our collected data include all cases $0 < a \leq 10{,}000$ that generate a set of three pairwise relatively prime numbers. Few sample data points are provided in Table 1, where some small and large values of a are chosen. We observed that $L \leq f(S) \leq U$ for $a \neq 1, 3$. Additionally, Figure 1 includes the graphs of the Frobenius number, the lower bound, and the upper bound as functions of $(xyz)^{\frac{1}{2}}$, as is done in [2]. The graph shows that the Frobenius number is closer to the upper bound than the lower bound.

Table 1: Frobenius Number, Upper & Lower Bounds Data for Arithmetic Sequence

$x = a$	$y = a + 1$	$z = a + 4$	L	$f(S)$	U	$(xyz)^{\frac{1}{2}}$
1	2	5	-2.52	DNE	-3.78	3.16
3	4	7	1.87	5	1.95	9.17
7	8	11	16.99	20	29.40	24.82
9	10	13	27.25	34	50.72	34.21
2,145	2,146	2,149	165,829.08	1,154,008	1,759,835.29	99,459.60
2,149	2,150	2,153	166,298.81	1,158,309	1,765,999.32	99,737.72
2,151	2,156	2,159	166,533.83	1,159,388	1,769,085.10	99,876.88
9,993	9,994	9,997	1,700,681.31	24,982,498.00	31,561,176.35	999,200.09
9,997	9,998	10,001	1,701,708.37	25,002,495.00	31,584,874.51	999,799.99
9,999	10,000	10,003	1,702,221.98	25,007,498.00	31,596,726.71	1,000,099.98

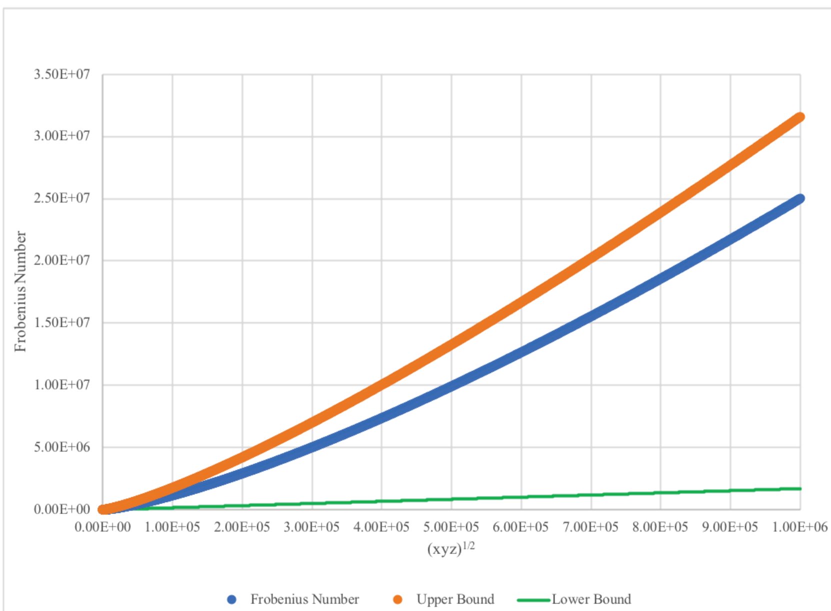

Figure 1: Comparison between Frobenius number in (3) and the bounds in (2) and (1).

3 Numerical Semigroups from Geometric Sequences

Let $a, b \in \mathbb{N}$ such that $\gcd(a, b) = 1$. Let S be the numerical semigroup generated by the geometric sequence $G_k = \{a^k, a^{k-1}b, \ldots, b^k\}$ for some natural number k.

Tripathi [7, Theorem 1] provided a formula for the Frobenius number as follows. If σ_k denotes the sum of integers in G_k, then $f(S) = \sigma_{k+1} - \sigma_k - a^{k+1} - b^{k+1}$. For $k = 2$, $\sigma_3 = a^3 + a^2b + ab^2 + b^3$ and $\sigma_2 = a^2 + ab + b^2$ and thus

$$f(S) = a^2b + ab^2 - a^2 - b^2 - ab. \tag{4}$$

This formula is used to test the bounds given in equations (1) and (2). However, it is worth noting that the elements of the generating set $\{a^2, ab, b^2\}$ are not pairwise relatively prime anymore. Our collected data include all pairs (a, b) such that $1 < a, b \leq 10,000$ and $\gcd(a, b) = 1$. Few sample data points are provided in Table 2, which is ordered by the values of b. The Frobenius number is smaller than the conjectured upper bound except for when $a = 2$ and $b = 3$ (or vice versa). Figure 2 shows that the upper bound grows at a much faster rate than the Frobenius number and the Frobenius number closely hugs the lower bound.

Table 2: Frobenius Number, Upper & Lower Bounds Data for Geometric Sequence

$x = a^2$	$y = ab$	$z = b^2$	L	$f(S)$	U	$(xyz)^{\frac{1}{2}}$
4	6	9	6.46	11	9.78	14.70
9	12	16	35	47	68.55	41.57
25	30	36	193.60	239	497.30	164.32
16	28	49	163.62	215	423.92	148.16
81	549	3,721	17,929.20	34,079	132,641.93	12,863.48
49	448	4,096	11,830.95	27,215	88,979.08	9,482.37
36	462	5,929	10,772.81	31,919	92,702.68	7,554.25
10,000	9,300	8,649	1,525,457.26	1,766,951	27,571,825.17	896,859.52
10,000	9,700	9,409	1625587.04	1,881,791	29,838,262.35	955,339.21
10,000	9,900	9,801	1676434.11	1940399	31002743.68	985,037.56

4 Numerical Semigroups from Compound Sequences

Let $(a_1, a_2, \ldots, a_k), (b_1, b_2, \ldots, b_k) \in \mathbb{N}^k$, for some $k \in \mathbb{N}$, such that $\gcd(a_i, b_j) = 1$ for $i \geq j$. Let $g_0 = \prod_{i=1}^{k} a_i$ and $g_i = g_{i-1}b_i/a_i = b_1 \cdots b_i a_{i+1} \cdots a_k$ for $1 \leq i \leq k$. Then the sequence $G = (g_i)_{i=0}^{k}$ is called a compound sequence. If S is the numerical semigroup generated by a compound sequence G, the authors in [9] gave the following formula for the Frobenius number

$$f(S) = -g_0 + \sum_{i=1}^{k} g_i(a_i - 1). \tag{5}$$

However, this formula is not an explicit formula. When $k = 2$, a compound sequence will be of the form $G = \{a_1a_2, a_2b_1, b_1b_2\}$ such that $\gcd(a_1, b_1) = \gcd(a_2, b_1) = 1$. Note that the definition of a compound sequence does not

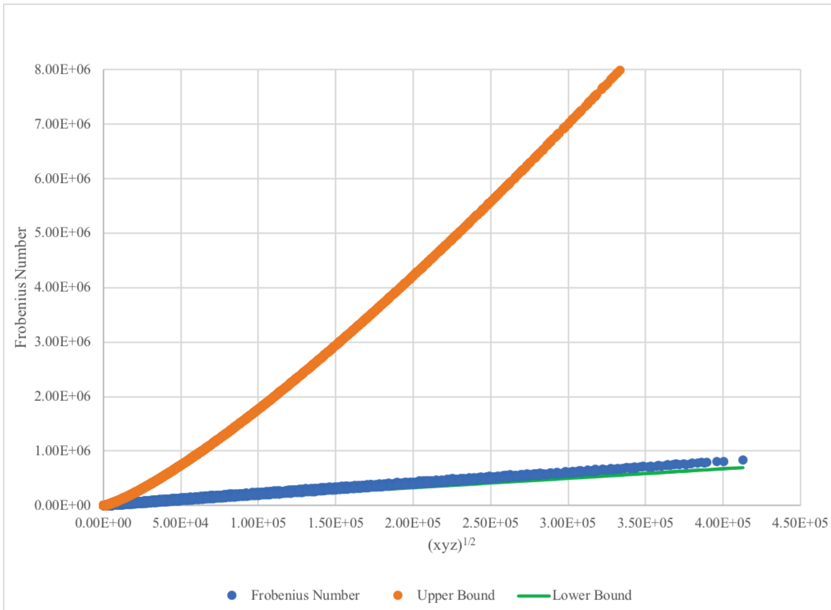

Figure 2: Comparison between Frobenius number in (4) and the bounds in (2) and (1).

necessitate that $\gcd(a_2, b_2) = 1$. Additionally, the elements of the generating set G are not pairwise relatively prime anymore. In the case $k = 2$, the above formula can be expressed explicitly as

$$f(S) = a_1a_2b_1 + a_2b_1b_2 - (a_1a_2 + a_2b_1 + b_1b_2).$$

The software GAP is used to calculate the Frobenius number in this case for $1 < a_1, a_2, b_1, b_2 \le 10$. Except for when $a_1a_2 = 4$, $a_2b_1 = 6$, $b_1b_2 = 9$, the conjectured upper bound was larger than the Frobenius number. Few sample data points are provided in Table 3. Additionally, Figure 3 shows that the Frobenius number closely hugs the lower bound more than the upper bound.

Table 3: Frobenius Number, Upper & Lower Bounds Data for Compound Sequence

(a_1, a_2, b_1, b_2)	$x = a_1a_2$	$y = a_2b_1$	$z = b_1b_2$	L	$f(S)$	U	$(xyz)^{\frac{1}{2}}$
(2,2,3,3)	4	6	9	6.46	11	9.78	14.70
(3,2,5,3)	6	10	15	20.96	29	39.21	30
(2,4,5,5)	8	20	25	56.54	87	125.36	63.25
(6,2,7,3)	12	14	21	55.88	79	117.89	59.40
(8,5,9,6)	40	45	54	401	491	1171.06	311.77
(5,9,7,10)	45	63	70	593.59	767	1868.60	445.48
(9,9,10,10)	81	90	100	1207.85	1439	4344.35	853.81

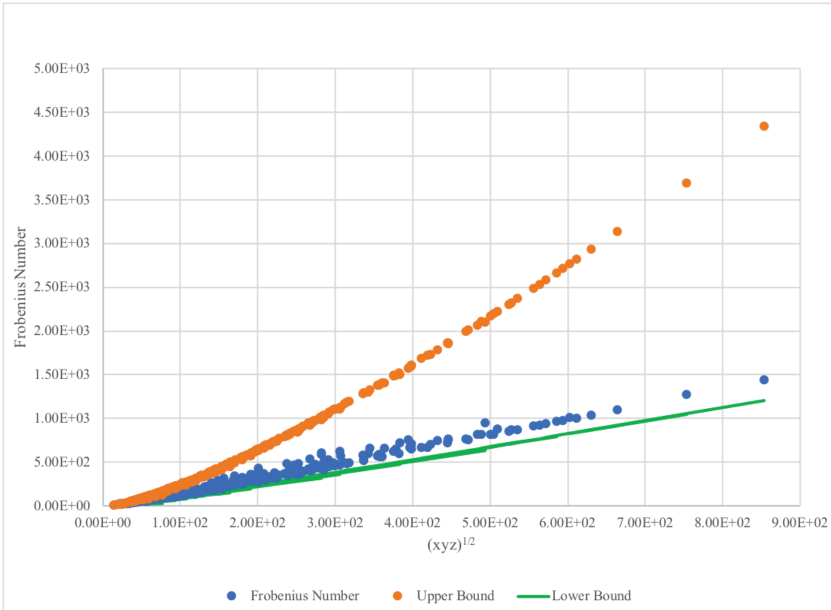

Figure 3: Comparison between Frobenius number in (5) and the bounds in (2) and (1).

5 Conclusion

As noticed, the Frobenius number in the case of arithmetic sequences was closer to the upper bound than the lower bound. For geometric sequences, the upper bound grew at a much faster rate than the Frobenius number, which closely hugged the lower bound. In the case of compound sequences, the Frobenius number also closely hugged the lower bound more than the upper bound. We do not have a good intuition on whether the Frobenius number will be closer to the upper bound or lower bound for most generating sets with three elements. We hope to investigate this further in future research.

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