A Derivation of the Navier-Stokes Equations

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Introduction

The Navier-Stokes equations are a set of second-order partial differential equations relating first and second derivatives of fluid velocity, which is represented as a smooth vector field. While simple in principle, they are enormously difficult to solve; in fact, no proof has yet been found guaranteeing even the existence of a smooth solution in just three dimensions. Annoying as this is to mathematicians, the consequences of a lack of solution are far-reaching: the Navier-Stokes equations govern continuum phenomena in all areas of science, from basic hydrodynamical applications to even cosmology.

Preliminaries

To understand and appreciate the Navier-Stokes equations, one must first be familiar with some of the basic concepts of fluid dynamics. We begin with the distinction between *intensive* and *extensive properties*.

An intensive property of a fluid is a property whose value does not depend upon the volume of measurement. For instance, pressure, density, momentum, and velocity are intensive properties. Mass, volume, and surface area are not. An extensive property of a fluid is a property whose value does depend upon the volume of measurement. (That is, it is a property that is not intensive.) Intensive properties are useful to us because they can be evaluated meaningfully and generally on differential elements.

The second concept we require is the *continuum hypothesis* (not related to set theory). Because we are dealing with a macroscopic fluid, we ignore that the fluid is composed of zillions of discrete particles and thus assume that the properties of the fluid may be described by continuous (actually, often C^∞) functions.

As a matter of fact, if we "zoom in" far enough, we find that intensive properties become ill-defined or discontinuous; since temperature, for instance, is the sum of the kinetic energies of the particles of a material, at the atomic scale temperature becomes a discrete function of position. But in the macroscopic limit, because of the relatively high particle density, we may, and do, regard temperature (and other intensive properties) as continuous functions.

The continuum hypothesis is relevant not just to continuum mechanics, but to all macroscopic physics that deal with continuous systems, including electrodynamics, thermodynamics, materials science, and hydrodynamics.

The Navier-Stokes equations deal chiefly with the relationship between the intensive properties pressure and velocity. Pressure p may be considered a C^∞ function $p : \mathbb{R}^n \times [0, \infty) \to (0, \infty)$, and velocity u a C^∞ vector field over $\mathbb{R}^n$ that varies with time. Generally, for physically applicable situations, we take $n = 2$ or 3.

We require a formulation of several conservation laws. Conservation laws are essential to physics; they permit us to use the constancy of various quantities to relate systems at different points in time. We shall need the law of conservation of mass and the law of conservation of momentum. In particular, the law of conservation of mass states that, given an isolated system, the amount of matter present in the system remains constant over time. Alternatively, mass may be neither created nor destroyed.

Let us apply the law of conservation of mass to a fluid. Consider an arbitrary volume V. Let dA be a surface element on the surface ∂V of V. On dA, the density ρ of the fluid is constant, as is the velocity u. If η_A is the outward unit normal to dA, then in a time dt, the volume of fluid that flows through dA is $u \cdot \eta_A dAdt$, so the mass that leaves through dA is

$$dM = \rho u \cdot \eta_A dAdt, \text{ or } \frac{dM}{dt} = \rho u \cdot \eta_A dA.$$

Integrating over the surface, we have the entire mass flux across ∂V equal to the rate of change of the mass within V:

$$\frac{dM}{dt} = -\iint_{\partial V} \rho u \cdot \eta_A dA = -\iiint_V (\nabla \cdot (\rho u)) d\tau,$$

using the divergence theorem.

But $M = \iiint_V \rho d\tau$, so $dM/dt = \iiint_V d\rho/dt d\tau$. This implies

$$0 = \iiint_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) \right] d\tau.$$

This is true for every $V \subseteq \mathbb{R}^n$, so we have that everywhere

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0.$$

This is known as the *mass continuity* equation; it describes the movement of mass through a continuous fluid.

When dealing with fluids, it is often useful to assume *incompressibility*: namely, that the density of a fluid is constant. In other words, given a container, it is impossible to ever pack more fluid into it or take fluid out without changing the volume. This is equivalent to saying that $\nabla \cdot u = 0$; using the mass continuity equation, if density is constant then we have $\nabla \cdot (\rho u) = \rho \nabla \cdot u = 0$; $\rho > 0$, so $\nabla \cdot u = 0$.

Stress and *body forces* are the two other important concepts we shall need. Body forces are, generally, forces per unit volume. They may be characterized by long-range bulk forces, such as gravity or the electromagnetic forces, and internal forces, which are caused by internal stresses induced by viscosity. A stress is merely an internal force acting on an imaginary surface within a body; it is described in units of force per unit area. The simplest example is the stress within a body subject to an axial uniform external force; for example, a pillar. The stress then is merely the total load divided by the cross-sectional area of the pillar. More generally, stresses vary from point to point within a body, and at a point are different for different planar slices through that point.

For a body, the stress at a point acting on a planar surface can be decomposed twice: the stress vector, into n components, and the surface into n planes each perpendicular to an axis. The i^{th} component of the stress T on a surface with unit normal $n = (n_j)$ can therefore be described by $T_i = \sigma_{ij} n_j$, where σ_{ij} is a second-degree tensor known as the *stress tensor*.

Finally, there are several conditions that must be imposed on solutions of the Navier-Stokes equations over $\mathbb{R}^n$ in order to ensure physical realism. They must not blow up as one moves to infinity, so the initial condition – a divergence-free C^∞ vector field $u^0(x)$ – must be either spatially periodic or be bounded on all of $\mathbb{R}^n$. Moreover, the kinetic energy of the fluid, which depends on the square of the speed, must be bounded: $\int |u(x,t)|^2 d\tau < C$ for all $t > 0$ and some positive constant C.

Derivation

The derivation of the Navier-Stokes can be broken down into two steps: the derivation of the *Cauchy momentum equation*, an equation governing momentum transport analogous to the mass transport equation derived above; and the linking of the stress tensor to the rate-of-strain tensor in order to simplify the Cauchy momentum equation.

Cauchy Momentum Equation

We consider an incompressible, viscous fluid filling $\mathbb{R}^n$ subject to an external body force f described as a time-variant vector field $f : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$. Let f_i denote the components of force. We begin the derivation of the Navier-Stokes equations by first deriving the Cauchy momentum equation.

Consider a volume element $d\tau$ in $\mathbb{R}^n$. The total body force acting in the x_i direction on $d\tau$ is due to f_i and to forces caused by the stress tensor σ_{ij}. Let us consider these stress forces. At x, the component σ_{ij} of the stress tensor represents the force per unit area in the x_i direction acting at a point on a plane cut through $\mathbb{R}^n$ normal to the x_j direction. There are n components of the stress tensor pointing in the x_i direction: $\sigma_{ij}, j \in \{1, \ldots, n\}$. Each of these stresses varies over the element $d\tau$ in the x_j direction by some small amount; call it $\partial\sigma_{ij}$. Since $d\tau$ has side lengths of dx_i, the rate of stress variation is thus $\partial\sigma_{ij}/\partial x_j$. This represents a body force on $d\tau$.

Therefore, in the x_i direction, the total force (body force multiplied by volume) on $d\tau$ is:

$$F_i = f_i d\tau + \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} d\tau.$$

The momentum of the above differential volume element in the x_i direction is $P_i = \rho u_i d\tau$; we use Newton's second law, $F = dP/dt$, and differentiate momentum, using the chain rule and noting that each x_i is a function of time and that $u_j = dx_j/dt$:

$$\frac{dP_i}{dt} = \frac{d}{dt}(\rho u_i d\tau) = \rho \left(\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j} \right) d\tau.$$

Finally, equating the two expressions for total force on $d\tau$, we have:

$$f_i d\tau + \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} d\tau = \rho \left(\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j} \right) d\tau;$$

letting Ω be an arbitrary volume in $\mathbb{R}^n$ and integrating,

$$\int_\Omega \left(f_i + \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} \right) d\tau = \int_\Omega \rho \left(\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j} \right) d\tau,$$

or

$$\int_\Omega \left[\left(f_i + \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} \right) - \rho \left(\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j} \right) \right] d\tau = 0.$$

Since the integrand is continuous and Ω is arbitrary, we have the Cauchy momentum equation:

$$f_i + \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} = \rho \left(\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j} \right).$$

Stress and Rate of Strain

As it stands, the problem is intractable: we have no idea what σ_{ij} is! But, turning to physical intuition, we have motivation for further assumptions: the stresses on the volume element should be related to the velocity gradient. For example, consider a simple stream of water where the flow is unidirectional and the speed of flow is related to distance from the bottom. If we consider a volume element at some height y from the bottom and follow it as it flows, we will notice it begin to shear as the top flows more quickly than the bottom.

This gives some physical basis to the definition of a *Newtonian fluid*: one where the stress on a volume element is linearly related to the velocity gradient. In the simple example above, we express this relationship as

$$\sigma = \mu \frac{du}{dy},$$

where μ is a proportionality constant known as *viscosity*.

More generally, consider a fluid element $d\tau$ in $\mathbb{R}^n$. The stresses acting on it can be broken down into two components: a normal uniform stress, known as pressure, which is the average of all of the normal stresses

$$p = -\frac{\sum \sigma_{ii}}{n}$$

(it is negative by convention); and a *deviatoric stress* $\tau_{ij} = \sigma_{ij} - p\delta_{ij}$, composed of the non-normal stress components and the deviation of the normal stresses from the pressure. The deviatoric stress is responsible for the deformation of the volume element, and is therefore related to the velocity gradient.

Now we concern ourselves with figuring out what the velocity gradient actually looks like. Consider a point $q = (q_i) \in \mathbb{R}^n$ and the associated velocity $(u_i(q))$. A small distance dq away from q, the velocity will have changed by some small quantity du given by the rule

$$du_i = dq \cdot \nabla u_i = \sum_j \frac{\partial u_i}{\partial x_j} dx_j.$$

We can rewrite this as

$$du_i = \sum_j \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) dx_j + \sum_j \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) dx_j,$$

or

$$du_i = \frac{1}{2} \sum_j (\epsilon_{ij} + \omega_{ij}) dx_j.$$

Here we set $\epsilon_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$ and $\omega_{ij} = (\partial u_i/\partial x_j - \partial u_j/\partial x_i)/2$; the former is known as the *rate of strain tensor* and the latter as the *vorticity tensor*. The rate of strain tensor is responsible for the deformation of the volume element $d\tau$; the vorticity tensor is responsible for the rotation of the element. This can be seen by an examination of the geometry of the element at the endpoints of a time interval $(t, t + dt)$.

Here, we apply the assumption that our fluid is Newtonian: the deviatoric stress tensor is proportional to the rate of strain tensor, or $\tau_{ij} = 2\mu\epsilon_{ij}$. This permits us to simplify the Cauchy momentum equation, and in particular the second term on the left side:

$$\begin{aligned}
\sum_j \frac{\partial \sigma_{ij}}{\partial x_j} &= \sum_j \frac{\partial}{\partial x_j}(\tau_{ij} + p\delta_{ij}) \\
&= \sum_j \frac{\partial}{\partial x_j}(2\mu\epsilon_{ij}) + \frac{\partial p}{\partial x_i} \\
&= \frac{\partial p}{\partial x_i} + \mu \sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\
&= \frac{\partial p}{\partial x_i} + \mu \sum_j \left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_j}{\partial x_j \partial x_i}\right) \\
&= \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i + \mu \frac{\partial}{\partial x_i} \sum_j \frac{\partial u_j}{\partial x_j} \\
&= \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i + \mu \frac{\partial}{\partial x_i} \nabla \cdot u = \nabla p + \mu \nabla^2 u_i,
\end{aligned}$$

using, variously, the continuity of second derivatives and the fact that our fluid is incompressible.

Setting density to 1 for convenience and substituting back into the momentum equation, we finally have the *Navier-Stokes equations for an incompressible Newtonian fluid*:

$$\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j} = f_i + \mu \nabla^2 u_i - \frac{\partial p}{\partial x_i}; \;\; \sum_j \frac{\partial u_j}{\partial x_j} = 0.$$

One can also write the above Navier-Stokes equations more succinctly and in more familiar form as follows.

$$\nabla \cdot u = 0 \qquad \textit{Mass Continuity Equation,}$$

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = f + \mu \nabla^2 u - \nabla p \qquad \textit{Cauchy Momentum Equations.}$$

Explanation and Significance

These equations are thought to govern the motion of all fluids; they thus are essential to the physics of continuum mechanics. Practically every area of science that makes macroscopic approximations at least touches on fluid dynamics – from cosmology and the motion of galaxies to nuclear fusion and the confinement of plasma, from electrodynamics and the flow of charges to aerospace engineering and the design of airplanes, the Navier-Stokes equations are vital to the understanding of the universe.

Yet, it is not known whether physically realistic smooth solutions are guaranteed to exist on all of $\mathbb{R}^3$! The Clay Mathematics Institute has identified this question as one of the top challenges facing mathematicians and physicists in the first several decades of the twenty-first century, and has thus issued a

$1,000,000 prize to any individual who can provide proof of, or counterexample to, the existence of smooth solutions to the form of the Navier-Stokes equations here derived.

More specifically, the challenge requires a mathematician to prove that any acceptable (periodic or bounded) initial condition to the Navier-Stokes equations has a solution, or that there is an acceptable (periodic or bounded) initial condition for which the Navier-Stokes equations do not admit a solution.

The solution of the Navier-Stokes equations is, as one might expect, very difficult, and will require new insight into the very foundations of partial differential equations. It is crucial, then, that we continue to advance in this area of mathematics and physics. For more discussion on the problem we refer the reader to the reference [1].

References

[1] The Millennium Prize Problems
`http://www.claymath.org/millennium/Navier-Stokes_Equations/`