An Introduction to Cwatsets and Graphs

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Cwatsets

The first cwatset was discovered in 1987 by Erich Friedman, then a student at Rose-Hulman Institute of Technology. Cwatsets were originally developed for statistical purposes, as a means of creating typical subsamples from sets that are not groups [1]. However, as the subject developed, it began to take on a much more algebraic flavor. Now, it is studied as much for its intrinsic value as an algebraic structure as for its statistical applications. Cwatsets have many parallels to group theory, and much of the study of the subject exploits group-theoretic results to yield more information about cwatsets. Interestingly enough, the subject has been developed almost exclusively by undergraduates, mostly students at Rose-Hulman and participants in programs at that school. Recently, relationships between cwatsets and graph theory have begun to arise. This paper is an introduction to the subject of cwatsets and explores some of these relationships with graph theory.[1]

To begin with an example, consider the subset $F = \{000, 110, 101\}$ of the additive group $\mathbb{Z}_2^3$. (We write the elements of $\mathbb{Z}_2^d$ as binary strings for convenience.) Clearly, F is not a subgroup of $\mathbb{Z}_2^3$, since it is not closed under addition.

[1] The introduction to cwatsets given in the first section is based on a monograph on cwatsets currently being prepared by Gary Sherman and Thomas Langley, of Rose-Hulman.

However, the set still has some interesting regularity. We notice that adding 110 to every element of the set yields the "coset" $F + 110 = \{110, 000, 011\}$. If we apply the permutation $(1,2)$ to each element of the set F in the natural way, we obtain the set $\{000,\ 110,\ 011\}$, denoted by $F^{(1,2)}$. (We will also use the notation c^σ to denote the application of the permutation σ to element $c \in \mathbb{Z}_2^d$.) Hence, we have $F + 110 = F^{(1,2)}$. Similarly, $F + 101 = F^{(1,3)}$. Thus, although F is not closed under addition, it is closed if we allow a permutation operation—a "twist" of sorts. Hence F is a "**c**losed **w**ith **a** **t**wist"-set, leading us to our definition of a cwatset.

Definition 1. A *cwatset* C is a nonempty subset of the additive group $\mathbb{Z}_2^d$ with the property that for every element $c \in C$ there is a permutation $\sigma \in S_d$ such that $C + c = C^\sigma$. We refer to d as the *degree* of the cwatset. Note that it is not necessary for every σ in S_d to satisfy $C + c = C^\sigma$ for some $c \in C$. Observe that every cwatset contains the identity element $\mathbf{0}$, because $c + c = \mathbf{0}$ for all $c \in C$.

We often denote cwatsets in matrix form, so that we can speak of the *columns* of a cwatset. For example, we can write F of the above example as:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Also, since $c + c = \mathbf{0}$ for all c in $\mathbb{Z}_2^d$, an equivalent definition of a cwatset is that for every c in C, there is a $\sigma \in S_d$ such that $C^\sigma + c = C$. In terms of individual elements of the set C, we are really saying that $b^\sigma + c \in C$ for each $b \in C$. We can think of any ordered pair (σ, c) of $S_d \times \mathbb{Z}_2^d$ as defining a function from $\mathbb{Z}_2^d$ to itself:

$$b^{(\sigma,c)} = b^\sigma + c, \quad \text{for } b \in \mathbb{Z}_2^d.$$

To see how various elements of $S_d \times \mathbb{Z}_2^d$ interact when applied to elements of $\mathbb{Z}_2^d$, we let a second element (τ, d) of $S_d \times \mathbb{Z}_2^d$ act on $b^{(\sigma,c)}$. Then we have:

$$(b^{(\sigma,c)})^{(\tau,d)} = (b^\sigma + c)^{(\tau,d)} = b^{\sigma\tau} + c^\tau + d = b^{(\sigma\tau, c^\tau + d)},$$

with the convention that function composition is read left-to-right. The relationship $(\sigma, c) \circ (\tau, d) = (\sigma\tau, c^\tau + d)$ is clearly recognizable as the group structure of the semidirect product $S_d \ltimes_\phi \mathbb{Z}_2^d$, given by the operation

$$(\sigma, c) * (\tau, b) = (\sigma\tau, c^\tau + b),$$

where $\phi : S_d \to \mathrm{Aut}(\mathbb{Z}_2^d)$ is the homomorphism given by $\phi(\tau)(c) = c^\tau$. Thus, we really have a (right) group action of $S_d \ltimes_\phi \mathbb{Z}_2^d$ on $\mathbb{Z}_2^d$. This group is what is known as the *wreath product* of $\mathbb{Z}_2$ by S_d, usually denoted by $S_d \wr \mathbb{Z}_2$.

Understanding how this group action interacts with a cwatset yields a wealth of information about the structure of the cwatset. What we are really interested in is the setwise stabilizer of a cwatset C, denoted by $\mathrm{Stab}(C)$, under this group action. If $c \in C$, there is some permutation σ in S_d such that $C^\sigma + c = C$, since C is a cwatset. Thus, writing $C^{(\sigma,c)}$ to denote the set obtained by letting (σ, c) act on each member of C, we have $C^{(\sigma,c)} = C$, so that $(\sigma, c) \in \mathrm{Stab}(C)$. Also, for any $(\sigma, c) \in \mathrm{Stab}(C)$, we know that $b^\sigma + c \in C$ for any $b \in C$. Specifically, then, $\mathbf{0}^\sigma + c = \mathbf{0} + c = c \in C$.

Definition 2. The stabilizer of a cwatset C under the group action of $S_d \wr \mathbb{Z}_2$ on $\mathbb{Z}_2^d$ consists of exactly those pairs (σ, c) such that $C^\sigma + c = C$. This subgroup of $S_d \wr \mathbb{Z}_2$ is denoted by M_C, and is often referred to as the *maximal covering group* of the cwatset, or simply the *M-group* of the cwatset.

For our cwatset F introduced above, we have:

$$M_F = \{(id, 000),\ ((2,3), 000),\ ((1,2), 110),\ ((1,2,3), 110),\ ((1,3), 101),\ ((1,3,2), 101)\}.$$

Examining this group closely produces several useful ideas that appear often in cwatset theory. First, we notice that, due to the way the semidirect product is constructed, the projection $\pi_1 : M_C \to S_d$ given by $\pi_1((\sigma, c)) = \sigma$ is in general a homomorphism. Its image is a group consisting of exactly those permutations $\sigma \in S_d$ for which there is some c in C with $C^\sigma + c = C$. We call this group the *permutation group* of C, and denote it as P_C. The kernel of this homomorphism is of course the subgroup made up of every element of M_C that has the identity in its first coordinate. Each coset in M_C of this subgroup is exactly $\pi_1^{-1}(\sigma)$ for some $\sigma \in P_C$, and so contains exactly those elements of M_C that have σ as their first coordinate. (In our example, π_1 is coincidentally an isomorphism between M_F and S_3.)

Although the surjective projection into the second coordinate, denoted by π_2, is not a homomorphism, the set $H = \{(\sigma, c) \in M_C | c = \mathbf{0}\}$ is another subgroup of M_C, since $(\sigma, \mathbf{0}) * (\tau, c) = (\sigma\tau, \mathbf{0}^\tau + c) = (\sigma\tau, c)$. Also, if (τ, c) and (μ, c) are elements of M_C, then

$$C^{\mu\tau^{-1}} = (C^\mu)^{\tau^{-1}} = (C + c)^{\tau^{-1}} = (C^\tau)^{\tau^{-1}} = C,$$

so that $(\mu\tau^{-1}, \mathbf{0}) \in M_C$. Thus, we know that $(\mu\tau^{-1}, \mathbf{0}) * (\tau, c) = (\mu, c) \in H \cdot (\tau, c)$. These computations show that the right cosets of the subgroup H partition M_C exactly by the binary components of the elements of the M-group. Thus each element of C appears the same number of times in M_C. We obtain a result for cwatsets much like Lagrange's theorem for groups: for any cwatset C, $|C|$ divides $|M_C|$, which in turn divides $|S_d \wr \mathbb{Z}_2| = d! \cdot 2^d$.

Theorem 3. The cardinality of a cwatset C of degree d divides $d! \cdot 2^d$.

Another concept that will be useful is the idea of *equivalent* cwatsets. These are two cwatsets that can be converted to each other by a permutation of their columns. Using our previous example,

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

is equivalent to F. These two cwatsets have essentially the same structure, as evidenced by the fact that their M-groups are conjugates of one another in the wreath product by the element ((1,2),000). This definition of equivalence is really just a reflection of the commutativity of the cartesian product inherent in the construction of $\mathbb{Z}_2^d$.

Recall that the image of M_C under π_2 is exactly C. Any subgroup G of $S_d \wr \mathbb{Z}_2$ for which $\pi_2(G) = C$ is called a *covering group* for C. The fact that M_C is the largest such group is what motivates us to call it the maximal covering group for C. Interestingly enough, for any subgroup G of $S_d \wr \mathbb{Z}_2$, the set $\pi_2(G)$ is always a cwatset.

We also note that the only other covering group for F is the subgroup

$$K = \{(id, 000),\ \ ((1,2,3), 110),\ \ ((1,3,2), 101)\}.$$

Since this group has order 3, it is of course cyclic. In terms of cwatsets, this means that the entire cwatset F can be generated by a single binary element and one of its associated permutations. That is,

$$110^{((1,2,3),110)} = 110^{(1,2,3)} + 110 = 011 + 110 = 101$$

$$101^{((1,2,3),110)} = 101^{(1,2,3)} + 110 = 110 + 110 = 000$$

$$000^{((1,2,3),110)} = 000^{(1,2,3)} + 110 = 000 + 110 = 110.$$

It seems logical, then, to make the following definition.

Definition 4. A cwatset which has a cyclic covering group is called a *cyclic cwatset.*

An interesting example to illustrate this concept is the cwatset

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is also a group. Clearly, it is the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, so it is not a cyclic group. However, it is a cyclic cwatset, with a (maximal) cyclic covering group generated by $((1,2), 10)$.

Finally, we examine the way that elements of M_F act on F under the group action. For example, as calculated above,

$$000^{((1,2,3),110)} = 110$$

$$110^{((1,2,3),110)} = 101$$

$$101^{((1,2,3),110)} = 000,$$

so that $((1,2,3), 110)$ permutes the elements of F cyclically. If we assign 000 the label 1, 110 the label 2, and 101 the label 3, then $((1,2,3), 110)$ induces the permutation $(1,2,3)$ on the elements of F. The map ϕ taking each element of M_F to this permutation it induces on the elements of F is actually a homomorphism. The image of this homomorphism is denoted by L_F, and is typically called the L-group of the cwatset. This group encapsulates the most important structure of the cwatset, since it tells us exactly the ways in which elements can be permuted by applying one single element to each element of the cwatset. The comparison to Cayley's theorem of elementary group theory is clear—just as every group can be expressed as a permutation group, so every cwatset acts in some ways like a group of permutations on its elements.

We end our introduction to cwatsets with a final concept that will be useful later.

Definition 5. The *weight* of a column of a cwatset is simply the number of 1's appearing in that column. A cwatset of cardinality n is said to be a *perfect cwatset* if all of its columns have weight k or $n-k$ for some $k \leqslant n/2$.

For example, F is a perfect cwatset of cardinality $n = 3$ with $k = 1$.

These are some of the main concepts of the basic theory of cwatsets. We now turn to the way cwatsets can be produced from graphs.

Definition 6. A *graph* Γ is a 2-tuple consisting of a vertex set V and a (multi) set E of edges, each of which is an ordered or unordered pair of elements of V. In a *simple graph*, each edge e of E is a set consisting of exactly two distinct elements of V while no edge appears more than once. Two graphs are *isomorphic* if the vertices can be relabeled such that the graphs are the same.

This is a rather formalistic definition of a graph. From a practical standpoint, a simple graph is merely a collection of points (the vertices) with some line segments (the edges) connecting various pairs of vertices.

A concept that comes up often in graph theory is that of a complementary graph. The *complementary* graph Γ' of a graph Γ, is the graph on the same vertices as Γ, but which contains exactly those edges which are absent from Γ. Also, a *self-complementary graph* is one that is isomorphic to its complementary graph.

In 2001, Nancy-Elizabeth Bush and Paul Isihara of Wheaton College created a method, which relates the theory of graphs and cwatsets [2]. They presented a construction translating an isomorphism class of simple graphs into a cwatset. We will illustrate this construction through the following example:

Example 7. Consider the isomorphism class of the graph

$$\Gamma = (\{1, 2, 3\}, \{\{1, 2\}\})$$

in Figure 1. We can assign a value $y \in \mathbb{Z}_2^3$ to each graph to represent which edges exist in that graph. The edges that can appear in these graphs are $\{1, 2\}$, which corresponds to the first column of y; $\{1, 3\}$, which corresponds to the second column; and $\{2, 3\}$, represented by the third column. Of course, a "1" denotes that the corresponding edge is present in the graph. This has been done for each graph in the figure and written into the middle of the graph. We then find the sums of each pair of graph words, which are shown by the arrows between the graphs in the picture. Next, one of the graphs is singled out and we create a set consisting of the pairwise sums of this word and the other graph words. (Notice that $y + y = \mathbf{0}$, so $\mathbf{0}$ will always be an element of this set.) This set is our cwatset. If we use the top graph Γ in the picture as our base graph, the cwatset produced will be F. Also notice that since each separate y-value is merely a permutation of the others, the sets of sums are also permutations of each other. Therefore, no matter which graph we use to find our set, we will always obtain equivalent cwatsets.

Using this method of construction, Bush and Isihara proved that any isomorphism class of simple graphs on v vertices will always result in a cwatset. The cardinality of this cwatset is equal to the size of the isomorphism class of the graph and its degree is $d = \binom{v}{2}$. They also proved that no matter which graph from the isomorphism class is used, the cwatsets produced by this process will all be equivalent.

We note that, given the isomorphism class of a graph Γ, the isomorphism class of the complementary graph Γ' will produce the exact same cwatset as Γ. It is from this point that we continued the research to discover other facts regarding the connection between cwatsets and graphs.

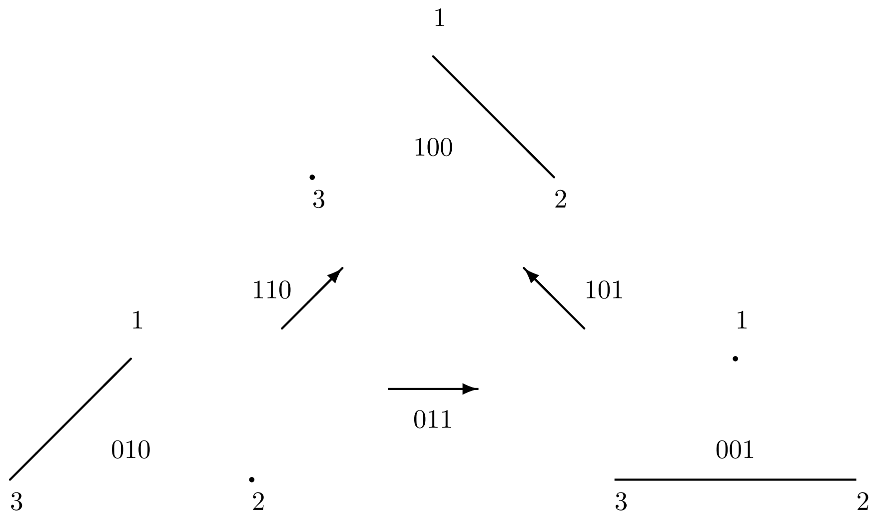


Figure 1

Graphs and Cwatsets

Proposition 8. *Cwatsets obtained from graphs contain only even-weight elements.*

Proof. First, note that the weight of the sum of two elements of $\mathbb{Z}_2^d$ is equivalent to addition of the individual weights modulo two (i.e. $w(a + b) \equiv w(a) + w(b) \bmod 2$). Now, the words associated with the graphs in an isomorphism class all have the same weight, since the graphs all have the same number of edges. Since the words in the associated cwatset are obtained by adding two graph words, the cwatset words must have even weight. □

Proposition 9. *A graph cwatset contains* **1** *iff the isomorphism class is self-complementary.*

Proof. Assume that a given graph cwatset contains $\mathbf{1}$. Then for y, the word representation of the original graph, there exists a word y' of another graph such that $y + y' = \mathbf{1}$. This implies that y' is the complement of y. Therefore, the isomorphism class contains the complement to the original graph. Similarly, assuming that the isomorphism class is self-complementary, the previous argument works again in reverse. □

Proposition 10. *Graph cwatsets are perfect.*

Proof. First notice that in a given isomorphism class, each edge appears in the same number of graphs. Call this number k. If a given edge appears in the original graph used to build the cwatset, then there are k zeroes in the column corresponding to that edge. Otherwise, if a given edge does not appear in the original graph, then there are k ones in the corresponding column. Therefore, the column weights will be either k or $n-k$. □

For the next proposition, the following definitions are needed.

Definition 11. A *directed graph* is a graph where an edge is defined by an ordered pair (a,b), so that the edge is thought of as directed from a to b.

Definition 12. A *multigraph* is a graph in which the edge set E is a multiset, so that multiple edges connecting the same two vertices may appear.

Definition 13. A *pseudograph* is a graph in which the edge set E is a multiset that may contain one point sets, so that multiple edges may appear in the graph, and edges that are incident to only one point, called loops, are allowed to appear in the graph.

We can extend the scheme used for numbering the edges of simple graphs to any directed graph, pseudograph, or multigraph as follows. In a directed graph on v vertices, there are a finite number of possible edges, so we can assign each possible edge a column in $\mathbb{Z}_2^n$, where $n = 2\binom{v}{2}$ is the total number of potential edges, just as was done for simple graphs. In the case of multigraphs, we find the edge that appears in E with the greatest multiplicity m and assign each edge that many columns in $\mathbb{Z}_2^n$, where $n = m\binom{v}{2}$. In pseudographs, we do the same thing, but count loops as potential edges, so that $n = m\binom{v+1}{2}$. In this way, every edge that could potentially appear in any graph in the isomorphism class has a column devoted to it. Once we have established this correspondence, the proof of the following theorem follows exactly as it did for the case of simple graphs.

Proposition 14. *An isomorphism class of directed, multi, or pseudographs results in a cwatset.*

In fact, the construction of a cwatset from a graph leaves even more room for generalization. Inspecting the construction closely, we see that when we represent a graph with a binary string indicating which edges are present, we could just as easily represent this graph by a set whose elements are exactly the numbers corresponding (under our numbering scheme) to the edges present in the graph. For example, the graph words 100, 010, 001 of our illustrative example could be represented by the sets $\{1\}$, $\{2\}$, $\{3\}$. Then, the process of adding one binary word to each other word is mirrored by selecting one set and taking the symmetric difference of it and the other sets. For example, if $\{1\}$

represents our base graph, the resulting sets will be $\{\}$, $\{1,2\}$, $\{1,3\}$, which are easily translated back into the elements of F, 000, 110, and 101. This interpretation of the construction essentially changes the graph isomorphism class into a hypergraph.

Definition 15. A *hypergraph* is a 2-tuple consisting of a vertex set V and an edge set E, which can be any subset of the power set of V.

In our example, $V = \{1,2,3\}$ and $E = \{\{1\},\{2\},\{3\}\}$. Hypergraphs become much more difficult to draw clearly because the edges can connect multiple vertices, or in this case, just one vertex. However, a hypergraph does give us the advantage of representing the entire isomorphism class of a graph in a single object, a hypergraph on a number of vertices equal to the number of edges that could possibly be present in a graph. However, with this generalization, we also realize that it is not necessary to look at an isomorphism class of a given graph. All that is really required to create a cwatset is for the set of graphs to be the orbit of a single graph under some group of permutations of the edges. In terms of hypergraphs, then, we have the following theorem.

Theorem 16. *Given any subset e of $\{1,2,\ldots,d\}$ and any subgroup G of S_d, the orbit of e under G, when viewed as a hypergraph, results in a cwatset.*

The connection between graphs and hypergraphs gives us a number of advantages in the study of the cwatsets produced by the entire isomorphism class of a simple graph. There is a long-standing connection between cwatsets and hypergraphs, originally discovered by Julie Kerr [4]. She discovered a way to express a perfect cwatset with no half-weight columns as a hypergraph.

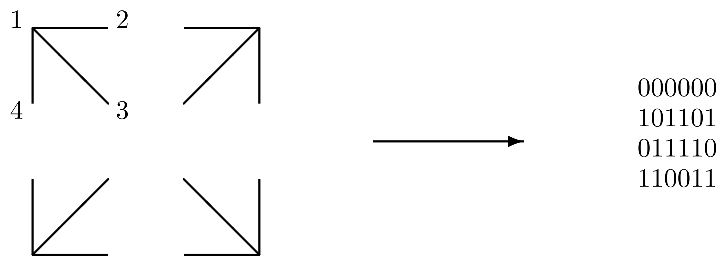
In brief, the conversion from cwatset to hypergraph depends on identifying the so-called "heavy" and "light" columns of the cwatset. A column is called "heavy" if its weight is more than half the cardinality of the cwatset, and "light" if its weight is less than half the cardinality. (The process is well-defined only if the cwatset has no columns whose weight is exactly half the cardinality of the cwatset, called *half-weight columns.*) Each element c of the cwatset generates a single edge of the hypergraph as follows. If c has a 0 in a heavy column, we include the vertex corresponding to that column in the resulting edge of the hypergraph. Similarly, if c has a 1 in a light column, we include that column's vertex in the edge. We demonstrate this process with the following cwatset, whose first and second columns are heavy, while the others are light.

$$\begin{array}{lcl} 000000 & \rightarrow & \{1,2\} \\ 111100 & \rightarrow & \{3,4\} \\ 100010 & \rightarrow & \{2,5\} \\ 111001 & \rightarrow & \{3,6\} \\ 010010 & \rightarrow & \{1,5\} \\ 110101 & \rightarrow & \{4,6\} \end{array}$$

It turns out that this hypergraph represents the orbit of a graph under a certain permutation of its edges, and that this set of graphs will produce the exact cwatset we started with. Thus, the graph and hypergraph representations of a graph cwatset are largely equivalent. The hypergraph representation has been studied the most, and has a number of useful results. This previous work was, and probably will continue to be, very helpful in the study of graph cwatsets.

Examples

With all this in mind, we set out to study the cwatsets generated by the entire isomorphism class of graphs on a small number of vertices, searching for various patterns. As suggested in [2], we hoped to classify the graphs that produce groups and/or cyclic cwatsets in order to develop a cwatset-based isomorphism test for graphs. Unfortunately, there is still only one example of a group cwatset, the one produced by $K_{1,3}$ as follows.



One obvious necessary condition for a graph to produce a group cwatset is that the isomorphism class of the graph has an appropriate cardinality, i.e. some power of 2. Another necessary condition is that the graph has half the number of potential edges. This must be the case since a group cwatset always has all half-weight columns, implying that half of its columns will be considered heavy in any hypergraph representation of it. Thus, half the edges of the graph representation will be present.

Among the graphs on 7 or fewer vertices, the only isomorphism class whose cardinality allows its cwatset to be a group is the graph cited above. In any case, it appears that groups are generated by graphs too rarely to be of much use as an isomorphism test.

To find graphs that produce cyclic cwatsets, we turn to another theoretical connection between a cwatset and its hypergraph. Just as every element of the M-group of a cwatset induces some permutation of the cwatset's elements, every element of the stabilizer of a hypergraph permutes its elements.

Definition 17. The group of permutations that represents the action of the various elements of the stabilizer on the hypergraph is called the *L-group of the hypergraph*, and is denoted L_H.

Theorem 18. *For any perfect cwatset C and a hypergraph representation H of it, $L_H \subseteq L_C$. (This assumes that the elements of H and C are numbered in a corresponding fashion.)*

Proof. Let σ be a permutation in the stabilizer of H. Label the elements of H as $h_1, h_2, \ldots, h_n$, with h_1 the representative of the zero word. The j-th element of the cwatset is denoted by c_j. We then note that the symmetric difference $h_1 \oplus h_i$ gives a set whose elements are exactly the columns in which the word represented by h_i has 1's. We refer to such a set as the set *describing* c_i, denoted $c_i \sim h_1 \oplus h_i$. Now, consider the set $h_1 \oplus h_1^\sigma$. This will be the set describing the element of the cwatset represented by h_1^σ. Call this element c. We claim that (σ, c) is an element of the M-group and that it induces the same permutation on the elements of C as σ does on H. Consider any cwatset word c_i, described by $h_i \oplus h_1$. Now,

$$\begin{aligned} c_i^\sigma + c &\sim (h_i \oplus h_1)^\sigma \oplus h_1 \oplus h_1^\sigma \\ &= h_i^\sigma \oplus h_1^\sigma \oplus h_1 \oplus h_1^\sigma \\ &= h_i^\sigma \oplus h_1. \end{aligned}$$

This last expression describes the word represented by h_i^σ, so that (σ, c) is indeed in the M-group, and the induced permutation on the words of the cwatset is the same as the induced permutation on the hypergraph. □

Corollary 19. *If there is an edge permutation that induces an n-cycle on the n elements of the hypergraph representation of a cwatset, then the cwatset is cyclic.*

This allows us to prove fairly directly that the graph with only one edge on any number of vertices always produces a cyclic cwatset. Also, the bipartite graph on v vertices $K_{1,v-1}$ will always produce a cyclic cwatset.

The final class of cyclic graph cwatsets comes from the bipartite graphs $K_{2,v-2}$. However, not all of these graphs actually produce cyclic cwatsets. Of those that we investigated, $K_{2,3}$ on 5 vertices and $K_{2,5}$ on 7 vertices are the only graphs that do not. (This includes $K_{2,1}$, which also falls into the category discussed above.) A logical conjecture regarding $K_{2,v-2}$ is that cyclic cwatsets result only when v is even, with the exception of $v = 3$. While these patterns suggest that the complete bipartite graphs tend to produce cyclic cwatsets relatively often, we note that $K_{3,3}$ and $K_{3,4}$ do not.

For a more detailed account of this research, we point the interested reader to the full technical report [3].

References

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