An invariant of metric spaces under bornologous equivalences

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We define bornologous equivalence between metric spaces, which is a more restrictive equivalence than coarse equivalence. It requires spaces to have the same cardinality. We define an invariant of metric spaces under bornologous equivalences. The invariant is essentially the number of ways that sequences can go to infinity in a space. This invariant is only for a certain class of spaces that we call sigma stable.

Introduction

A function $f : X \to Y$ between metric spaces is uniformly continuous if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $d(x, y) < \delta$, $d(f(x), f(y)) < \epsilon$. A dual concept is that of a function being bornologous. A function is (uniformly) bornologous if for every $N > 0$ there is an $M > 0$ such that if $d(x, y) \le N$, $d(f(x), f(y)) \le M$ [1]. While continuity is a small scale property, bornology is a large scale or coarse property.

Let us define two metric spaces X and Y to be bornologously equivalent if there are bornologous functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \equiv \mathrm{id}_X$ and $f \circ g \equiv \mathrm{id}_Y$. This equivalence defines a bornologous category of metric spaces.

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The coarse category of metric spaces is defined using coarse maps as morphisms. A map is coarse if it is bornologous and proper. A map is proper if inverse images of bounded sets are bounded. Two metric spaces X and Y are coarsely equivalent if there are coarse functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is close to id_X and $f \circ g$ is close to id_Y [1] (two functions $f, g : X \to Y$ are close if there is an $N > 0$ such that $d(f(x), g(x)) \leq N$).

The motivation behind the definition of coarse equivalence is that a large scale property should not depend on local properties of a space, including cardinality. The motivating example is the real numbers $\mathbb{R}$ and the integers $\mathbb{Z}$. These two spaces are coarsely equivalent but cannot be bornologously equivalent because they do not have the same cardinality.

It is not hard to see that if X and Y are bornologously equivalent then they are coarsely equivalent. We find it worthwhile to investigate this more restrictive category. We also hope that our techniques will generalize to the coarse category.

We recall three standard metrics on the cartesian product of two metric spaces. Suppose (X, d_X) and (Y, d_Y) are metric spaces. For $(x_1, y_1), (x_2, y_2) \in X \times Y$, set

$$d_{L_2}((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2} \text{ (the } L_2 \text{ metric)},$$

$$d_{TC}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2) \text{ (the taxicab metric), and}$$

$$d_M((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} \text{ (the max metric).}$$

It is left as an exercise that these metrics on $X \times Y$ are all bornologously equivalent. We will use the taxicab metric on subspaces of $\mathbb{R}^2$ and denote it simply as d.

Given a metric d on X, the associated standard bounded metric $\min\{d, 1\}$ is uniformly continuous equivalent to d. We define an analog for the large scale.

Definition 1. Suppose (X, d) is a metric space. For each pair $x, y \in X$, set

$$D(x, y) = \begin{cases} 0 & x = y \\ \max\{d(x, y), 1\} & x \neq y \end{cases}$$

D is the associated discrete metric on X.

It is easy to see that D is bornologously equivalent to d.

The following example illustrates the difference between the small scale and the large scale.

Example 2. Let the stairs S be the graph of the step function, $S = \{(x, \lfloor x \rfloor) : x \in \mathbb{R}\}$. (See Figure 1). We consider the natural function $f : \mathbb{R} \to S$ that sends x to $(x, \lfloor x \rfloor)$. Let us see that f is bornologous. Suppose $N > 0$ and $d(x, y) \leq N$. Then $d((x, \lfloor x \rfloor), (y, \lfloor y \rfloor)) = |x - y| + |\lfloor x \rfloor - \lfloor y \rfloor| \leq 2N + 1$.

Now consider the function $g : S \to \mathbb{R}$ that sends $(x, \lfloor x \rfloor)$ to x. Notice $g \circ f \equiv \mathrm{id}_X$ and $f \circ g \equiv \mathrm{id}_Y$. Let us see that g is bornologous. Suppose $N > 0$ and $d((x, \lfloor x \rfloor), (y, \lfloor y \rfloor)) \leq N$. Then $d(x, y) = |x - y| \leq N$ since $|x - y| + |\lfloor x \rfloor - \lfloor y \rfloor| \leq N$.

Therefore S and $\mathbb{R}$ are bornologously equivalent. In contrast these two spaces are not equivalent under uniform continuity. There are breaks in the graph of the step function which represent discontinuities. They do not have the same structure in the small scale.

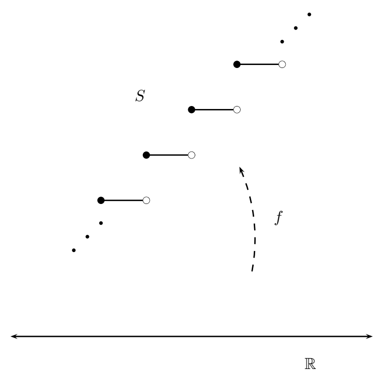

Figure 1: The stairs are bornologously equivalent to $\mathbb{R}$.

The following is an example of two spaces that should not be bornologously equivalent. It is the motivating example for our theory.

Example 3. Let the vase V be the union of the sets $A = \{(1, y) : y \geq 1\}, B = \{(x, 1) : -1 \leq x \leq 1\}$, and $C = \{(-1, y) : y \geq 1\}$. We will consider a natural, bijective function $f : V \to \mathbb{R}$ that flattens out the vase onto the real line and see that it is not bornologous. (See Figure 2). Define

$$f(x, y) = \begin{cases} y & \text{if } (x, y) \in A \\ x & \text{if } (x, y) \in B \\ -y & \text{if } (x, y) \in C \end{cases}.$$

Suppose $M > 0$. We want to show that there is $((x_1, y_1), (x_2, y_2)) \in V$ such that $d((x_1, y_1), (x_2, y_2)) \leq 2$ but $d((f(x_1, y_1), f(x_2, y_2)) > M$. Consider the points $(-1, M), (1, M) \in V$. Then $d((-1, M), (1, M)) = |-1 + (-1)| + |M - M| = 2$ but $d(f(-1, M), f(1, M)) = d(-M, M) = 2M$.

Even though the function in the previous example is not bornologous, in order to show that the two spaces are not bornologously equivalent we would need to show that every function is not a bornologous equivalence. This task is not feasible. We define an invariant that will be able to distinguish between these two spaces in the bornologous category. An invariant is a property of a space that is not changed under bornologous equivalence. Therefore if two spaces have a different invariant property, then they cannot be bornologously equivalent.

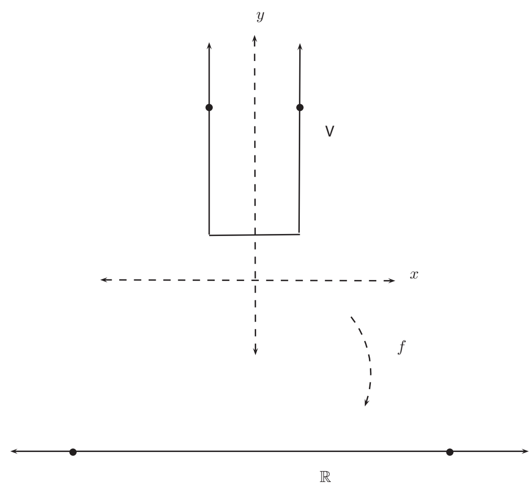

Figure 2: The vase should not be bornologously equivalent to $\mathbb{R}$.

Sequences

We try to distinguish between spaces in the bornologous category using sequences that go to infinity. We try to see how many different ways there are to go to infinity. In Example 3, $\mathbb{R}$ has two ways of going to infinity (to the right and to the left) but V really has only one since the two sides are of bounded distance from one another.

Definition 4. Given a metric space (X, d) and an $N > 0$, an N-chain in X is a finite list $x_0, \ldots, x_n$ of points in X such that $d(x_i, x_{i+1}) \leq N$ for all $i < n$. An N-sequence in X is an infinite list $x_0, x_1, \ldots$ such that $d(x_i, x_{i+1}) \leq N$ for all $i \geq 0$.

The following is a nice interpretation of a bornologous function. Given a bornologous function $f : X \to Y$ where if $d(x, y) \leq N$ then $d(f(x), f(y)) \leq M$, f sends N-chains in X to M-chains in Y and N-sequences in X to M-sequences in Y.

We are only interested in sequences that go to infinity.

Definition 5. Let X be a metric space with basepoint x_0 and $N > 0$. We consider an N-sequence to be based at x_0 if its first element is x_0. An N-sequence $x_0, x_1, x_2, \ldots$ goes to infinity, $(x_i) \to \infty$, if $\lim_{i \to \infty} d(x_i, x_0) = \infty$. Let $S_N(X, x_0)$ be the set of all N-sequences in X based at x_0 that go to infinity.

We want to consider an equivalence between sequences. A motivating example is that all sequences s in $\mathbb{R}$ with $\lim s = \infty$ should be equivalent.

Definition 6. Given $s, t \in S_N(X, x_0)$, define s and t to be related, $s \sim t$, if s is a subsequence of t or t is a subsequence of s. If t is a subsequence of s we say that s is a supersequence of t. Define s and t to be equivalent, $s \approx t$, if there is a finite list of elements $s_i \in S_N(X, x_0)$ such that $s \sim s_1 \sim s_2 \sim \cdots \sim s_n \sim t$. Let $[s]_N$ denote the equivalence class of s in $S_N(X, x_0)$ and $\sigma_N(X, x_0)$ be the set of equivalence classes.

Let us illustrate this definition by comparing it to the standard definition of a limit of a sequence in $\mathbb{R}$ equaling infinity.

Lemma 7. *Suppose $N > 0$ and (x_i) is an N-sequence in $\mathbb{R}$ based at 0. Then $(x_i) \to \infty$ if and only if $\lim x_i = \infty$ or $\lim x_i = -\infty$.*

Proof.
($\Leftarrow$) Suppose $M > 0$.
Case 1: $\lim x_i = \infty$. Then there is an $n > 0$ such that $x_i > M$ for all $i \geq n$. Then $d(x_0, x_i) = |x_i| = x_i > M$ for all $i \geq n$ since x_i is positive. Therefore $(x_i) \to \infty$.
Case 2: $\lim x_i = -\infty$. Then there is an $n > 0$ such that $x_i < -M$ for all $i \geq n$. Then $d(x_0, x_i) = |x_i| = -x_i > M$ for all $i \geq n$ since x_i is negative. Therefore $(x_i) \to \infty$.

($\Rightarrow$) Suppose $M > N$ and $(x_i) \to \infty$. Then there is $n > 0$ such that $|x_i| > M$ for all $i \geq n$.
Case 1: $x_n \geq 0$. Then $x_n > M$. Since $|x_n - x_{n+1}| \leq N \leq M$, $x_{n+1} > 0$. So, $x_{n+1} > M$. Similarly $x_{n+2} > M$. We continue to get $x_i > M$ for all $i \geq n$. Then $\lim x_i = \infty$.
Case 2: $x_n < 0$. Then $-x_n > M$ and $x_n < -M$. Since $|x_n - x_{n+1}| \leq N \leq M$, $x_{n+1} < 0$. So, $-x_{n+1} > M$. Similarly $-x_{n+2} > M$. We continue to get $x_i < -M$ for all $i \geq n$. Then $\lim x_i = -\infty$. $\square$

We can also see that a sequence whose limit is infinity cannot be equivalent to a sequence whose limit is negative infinity.

Lemma 8. *Suppose $(x_i), (y_i) \in S_N(\mathbb{R}, 0)$.*

1. *If $\lim x_i = \infty$ and $(x_i) \sim (y_i)$ then $\lim y_i = \infty$.*
2. *If $\lim x_i = -\infty$ and $(x_i) \sim (y_i)$ then $\lim y_i = -\infty$.*

Proof.

1. Suppose $\lim x_i = \infty$ and $(x_i) \sim (y_i)$. Suppose $M > 0$. Since $\lim x_i = \infty$, there is an $n > 0$ such that $x_i > M$ for all $i \geq n$.

   Case 1: Suppose y_i is a subsequence of x_i i.e., $y_i = x_{m_i}$ where (m_i) is an increasing sequence of natural numbers. Choose k so that $m_k \geq n$. Then for all $i \geq k$, $y_i = x_{m_i} > M$ since $m_i > m_k \geq n$. Therefore $\lim y_i = \infty$.

   Case 2: Suppose y_i is a supersequence of x_i. Suppose to the contrary that $\lim y_i \neq \infty$. Then by Lemma 7, $\lim y_i = -\infty$. Then by Part (2) Case 1 of this lemma, $\lim x_i = -\infty$, a contradiction. Therefore $\lim y_i = \infty$.

2. Suppose $\lim x_i = -\infty$ and $(x_i) \sim (y_i)$. Suppose $M < 0$. Since $\lim x_i = -\infty$, there is an $n > 0$ such that $x_i < M$ for all $i \geq n$.

   Case 1: Suppose y_i is a subsequence of x_i i.e., $y_i = x_{m_i}$ where (m_i) is an increasing sequence of natural numbers. Choose k so that $m_k \geq n$. Then for all $i \geq k$, $y_i = x_{m_i} < M$ since $m_i > m_k \geq n$. Therefore $\lim y_i = -\infty$.

   Case 2: Case 2 is similar to Part (1) Case 2.

□

The invariant

Given a metric space X we wish to determine the cardinality of $\sigma_N(X, x_0)$ and use it as an invariant. Now this cardinality obviously depends on the number N. To deal with this complication we have the following definition.

Definition 9. Let X be a metric space with basepoint x_0. For each integer $N > 0$ there is a function $\phi_N : \sigma_N(X, x_0) \to \sigma_{N+1}(X, x_0)$ that sends an equivalence class $[s]_N$ to the equivalence class $[s]_{N+1}$. Define X to be σ-stable if there is an integer $K > 0$ such that ϕ_N is a bijection for each $N \geq K$. If X is σ-stable define $\sigma(X, x_0)$ to be the cardinality (size) of $\sigma_K(X, x_0)$.

If $M \geq N \geq 0$ are integers, let ϕ_{NM} be the composition $\phi_M \circ \phi_{M-1} \circ \phi_{M-2} \circ \cdots \circ \phi_N$. It is identical to the function that sends an equivalence class $[s]_N \in S_N(X, x_0)$ to $[s]_M$. We can now prove our main theorem.

Theorem 10. *Suppose $f : X \to Y$ is a bornologous equivalence between metric spaces. Let x_0 be a basepoint of X and set $y_0 = f(x_0)$. Suppose X and Y are σ-stable. Then $\sigma(X, x_0) = \sigma(Y, y_0)$.*

Proof. Let K be the integer provided by the fact that X is σ-stable and K_1 be the integer provided by Y being σ-stable. Since f is bornologous there is an integer $M \geq K_1$ such that if $d(x, y) \leq K$, $d(f(x), f(y)) \leq M$. Since f^{-1} is bornologous there is an integer $L \geq K$ such that if $d(f(x), f(y)) \leq M$ then $d(x, y) \leq L$. Let $f_K : \sigma_K(X, x_0) \to \sigma_M(Y, y_0)$ be the function that sends an element $[s]_K$ to $[f(s)]_M$ and $f_M^{-1} : \sigma_M(Y, y_0) \to \sigma_L(X, x_0)$ be the function that sends $[f(s)]_M$ to $[s]_L$. We wish to show that f_K is a bijection.

Consider diagram (a) in Figure 3 below. This diagram commutes, i.e., $\phi_{KL} = f_M^{-1} \circ f_K$. Since ϕ_{KL} is one-to-one, f_K is one-to-one.

Now there is an integer $P \geq M$ such that if $d(x, y) \leq L$ then $d(f(x), f(y)) \leq P$. Let $f_L : \sigma_L(X, x_0) \to \sigma_P(Y, y_0)$ be the function that sends an equivalence class $[s]_L \in \sigma_L(X, x_0)$ to $[f(s)]_P$. Then we have the commutative diagram (b) in Figure 3, where φ_{MP} sends an equivalence class $[f(s)]_M \in \sigma_M(Y, y_0)$ to $[f(s)]_P$.

Since φ_{MP} is one-to-one, f_M^{-1} is one-to-one. Therefore, referring to the first diagram, f_K is onto since ϕ_{KL} is onto. □

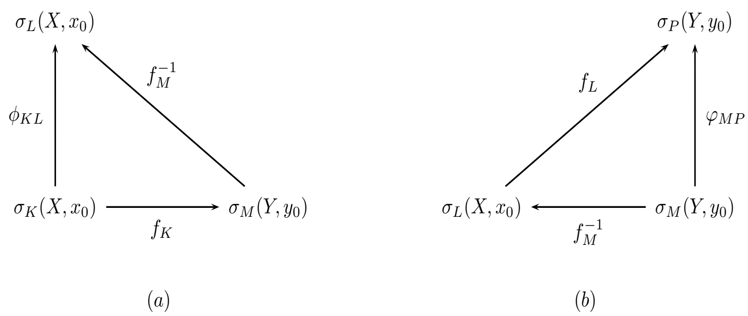

Figure 3: Commutative diagrams

We now see that in most cases $\sigma_N(X, x_0)$ is independent of choice of basepoint.

Definition 11. Let X be a metric space. Suppose $N > 0$ and $x, y \in X$. An N-chain from x to y is an N-chain $c_1, \ldots, c_n$ in X such that $c_1 = x$ and $c_n = y$. Let c denote the N-chain $c_1, \ldots, c_n$. If s is an N-sequence in X starting at y, we can define a new N-sequence $c * s$ starting at x by concatenating c and s. That is, if s is the sequence $s_1, s_2, \ldots$ then $c * s$ is the sequence $c_1, \ldots, c_{n-1}, s_1, s_2 \ldots$.

Proposition 12. *Suppose $N > 0$ and x_0 is a basepoint of a metric space X. Suppose $x_1 \in X$ and there is an N-chain c from x_1 to x_0. Then $\sigma_N(X, x_0)$ and $\sigma_N(X, x_1)$ have the same cardinality.*

Proof. Let $f : \sigma_N(X, x_0) \to \sigma_N(X, x_1)$ be the function that sends an element $[s] \in \sigma_N(X, x_0)$ to $[c * s]$. To see that f is well defined, first notice that if $s \in S_N(X, x_0)$ then $c * s \to \infty$. Next, notice if $s, t \in S_N(X, x_0)$ and s is a subsequence of t, then $c * s$ is a subsequence of $c * t$. Therefore f is well defined.

We can see that f is a bijection by noting that the inverse of f is given by $g : \sigma_N(X, x_1) \to \sigma_N(X, x_0)$ sending $[s]$ in $\sigma_N(X, x_1)$ to $[c^{-1} * s]$, where c^{-1} denotes the N-sequence from x_0 to x_1 that is the reverse of c. □

We now see that $\mathbb{R}$ and V are σ-stable and use the invariant to show that they are not bornologously equivalent.

Lemma 13. *Suppose $s \in S_N(\mathbb{R}, 0)$ with $\lim s = \infty$. Then there is $t \in S_N(\mathbb{R}, 0)$ with $s \sim t$ and t increasing.*

[1]

Proof. We will define an increasing sequence m_i of natural numbers and set $t_i = s_{m_i}$. Choose $m_1 = 1$. Since $\lim s_i = \infty$, there is an $m_2 > 1$ such that

$s_{m_2} > 0$. Then $t_2 = s_{m_2} > 0 = t_1$. There is an $m_3 > m_2$ such that $s_{m_3} > t_2$. Then $t_3 = s_{m_3} > t_2$. By using this technique, we continue to get an increasing subsequence t of s.

□

Theorem 14. *Suppose $N \geq 1$. Then $\sigma_N(\mathbb{R}, 0) = \{[(i)], [(-i)]\}$.*

Proof. We know that $\lim i = \infty$. By Lemma 7, $(i) \to \infty$. Similarly, we know that $\lim -i = -\infty$. By Lemma 7, $(-i) \to \infty$. So $[(i)], [(-i)] \in \sigma_N(\mathbb{R}, 0)$. Let us show $(i) \not\approx (-i)$. By Lemma 8, if $(i) \approx (-i)$, then $\lim -i = \infty$, a contradiction. Therefore $(i) \not\approx (-i)$.

Suppose $[s] \in \sigma_N(\mathbb{R}, 0)$. We need to show $[s] = [(i)]$ or $[s] = [(-i)]$. By Lemma 7, we have two cases.

Case 1: $\lim s = \infty$. We will to show $[s] = [(i)]$. We can assume s is increasing, by Lemma 13. We put s and (i) together to get a new N-sequence t, where $t \to \infty$. Since s and (i) are both increasing, we can use the order on the real line to arrange t so that it is an increasing supersequence of both s and (i). Since both s and (i) are subsequences of t, $s \approx (i)$.

Case 2: $\lim s = -\infty$. Similarly to Case 1, we can show that $s \approx (-i)$. □

Corollary 15. *The pointed metric space $(\mathbb{R}, 0)$ is σ-stable and $\sigma(\mathbb{R}, 0) = 2$.*

Proof. We know that $\sigma_N(\mathbb{R}, 0) = \{[(i)], [(-i)]\}$ for all $N \geq 1$. By definition, $\phi_N([(i)]_N) = [(i)]_{N+1}$. Also by definition, $\phi_N([(-i)]_N) = [(-i)]_{N+1}$. Therefore, ϕ_N is a 1-1 correspondence. □

Theorem 16. *Suppose $N \geq 2$. Then $\sigma_N(V, (1, 1)) = \{[((1, i))]\}$.*

Proof. We know $((1, i)) \to \infty$ so $[((1, i))] \in \sigma_N(V, (1, 1))$. Suppose $[s] \in \sigma_N(V, (1, 1))$ and $N \geq 2$. We want to show $s \approx ((1, i))$.

Let us create a supersequence of s. Every time a term (x, y) of s is not on A, add the following terms after it: $(1, y)$ and (x, y). Thus, we have a supersequence t of s that is an N-sequence that goes to infinity, so $s \sim t$.

Now create a subsequence u of t by eliminating all terms that are not on A. Then u is an N-sequence that goes to infinity so $t \sim u$. Thus we have a sequence that is equivalent to s that lies entirely on A. In a similar way to the proof of Theorem 14, we can show that $u \approx ((1, i))$. Therefore, $s \approx ((1, i))$. □

Corollary 17. *The pointed space $(V, (1, 1))$ is σ-stable and $\sigma(V, (1, 1)) = 1$.*

Since any two points in V can be joined by an N-chain, we can now conclude that V and $\mathbb{R}$ are not bornologously equivalent. We end by posing the problem of generalizing our theory to non σ-stable spaces and to the coarse category.

References

[1] J. Roe, Lectures on coarse geometry, University lecture series, 31. American Mathematical Society, Providence, RI, 2003.