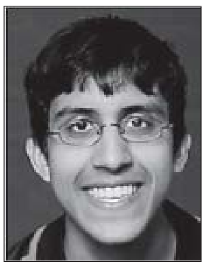
Characterization of a Family of Cubic Dynamical Systems

Shan Kothari



Shan Kothari is a second-year student majoring in anthropology and zoology at Michigan State University. This article was completed under the supervision of Dr. Aklilu Zeleke as part of a course offered in Fall 2010.

Abstract: Motivated by the fact that cubic maps have found potential applications to modeling of biological and physical processes, we examine a family of discrete, non-linear dynamical systems comprising one-parameter real variable cubic polynomials of a certain form. We examine and classify their fixed points and 2-cycles over various parametric domains. We also study their bifurcation diagrams and use a variety of techniques to analyze their chaotic behavior.

Introduction

The theory of discrete non-linear dynamical systems has been used to model many processes in economics, biology, and physics, among other fields. First-order difference equations, in particular, are often used to model the evolution of systems in which an assumption can be made that the state of the system at a given point in time can be derived from its state in the point immediately preceding. These situations may arise in such diverse fields as genetics (as a description of change in gene frequency), economics (toward describing temporal trends such as business cycles), and the social sciences (to study transmission of information) [6].

The logistic map, a simple system with one critical point described by the equation

$$f(x) = ax(1-x), \qquad 0 \le a \le 4$$

has undergone exhaustive analysis, as have other quadratic maps described by first-order difference equations. Guckenheimer, May, and others have analyzed the behavior of this map as one approach to modeling density-dependent population dynamics with non-overlapping generations [6].

While the dynamics of the family of first-order difference equations with quadratic maps has been thoroughly studied, the family of equations with cubic maps has not been studied as well [7], despite their possible utility in modeling scenarios in biology and physics. In part, this is because such equations are qualitatively more difficult to analyze. While quadratic maps appear quite simple, they show quite complicated dynamics [6]. In their most general form, cubics are much harder to analyze [7], as the general form of the roots of a cubic equation contains nested radicals and other complicated expressions. Nevertheless, the abundance of situations in which cubic maps may have relevance necessitates the study of the dynamics of these maps. Therefore, in this paper, we investigate several special cases of the general cubic map that are much simpler to analyze, and could lead to interesting insights in the study of more general cubic maps.

For a general background to the analysis of discrete, non-linear dynamical systems, see Devaney [3]. In addition to the principles in Devaney, we also used Bair and Haesbrock [1] to evaluate behavior near neutral fixed points.

In the present work, we consider the fixed points and 2-cycles of dynamical systems associated with seven related cubic functions $f : \mathbb{R} \to \mathbb{R}$. These functions are:

[f-1] $f(x) = x^3 + x^2 + cx$

[f-2] $f(x) = x^3 + cx^2 + x$

[f-3] $f(x) = cx^3 + x^2 + x$

[f-4] $f(x) = x^3 + cx$

[f-5] $f(x) = cx^3 + x$

[f-6] $f(x) = x^3 + cx^2$

[f-7] $f(x) = cx^3 + x^2$

where $c \in \mathbb{R}$. Together, these functions encompass all cubic polynomials in one parameter that have a fixed point at $x = 0$, and where the non-parametric coefficients are all 1. Future references to these functions will be made using their assigned numbers.

While the properties of cubic maps have not been as well-analyzed as those of quadratic maps such as the simple quadratic map $f(x) = x^2+c$ and the logistic map, some studies have been conducted to analyze the properties of particular cubic maps. Mukhamedov studied chaotic behavior in p-adic cubic dynamical systems of the form $f(x) = x^3+ax^2$ over $\mathbb{Q}_p$, classifying fixed points according to behavior and finding basins of attraction for attracting fixed points [9]. It was found that the structure of attractors in this cubic map is more complicated than that in quadratic dynamical systems, and that cubic p-adic dynamical systems have, in general, a more chaotic structure than quadratic ones [Mu]. in addition, the geometric structure of Siegel disks and basins of attraction was thoroughly investigated. The same function analyzed by Mukhamedov over $\mathbb{Q}_p$ is also analyzed in this paper over $\mathbb{R}$.

Skjolding et al. investigated the presence of various varieties of special bifurcations in cubic maps with fixed points at $x = 1$ and $x = -1$, as described by the equation

$$f(x) = ax^3 + bx^2 + (1 - a)x - b.$$

The different varieties of forward and reverse bifurcations were classified, and their appearance in dynamical systems of this form was analyzed in depth. It is claimed that dynamical systems with two critical points may model the effects on the ignition phenomenon in neural networks of inhibitory connections in the network combined with refractory mechanisms [10].

May describes the biological applications of dynamical systems such as the antisymmetric cubic map

$$f(x) = ax^3 + (1-a)x \qquad \text{over the interval } [-1,1],$$

in modeling problems in genetics and evolution in which selective forces are "frequency dependent," or affected by gene frequencies, so that a particular allele has a selective disadvantage when common, and an advantage when it is rare [7, 8]. While the results of this application were not conclusive, it was stated that the purely mathematical properties of this function and others with two critical points are of intrinsic interest, and merit further study as a natural progression from maps with one critical point. In particular, they are the least complicated among the first order difference equations to exhibit the phenomenon of alternative stable states.

Iriso and Peggs present the cubic map

$$\rho_{m+1} = a\rho_m + b\rho_m^2 + c\rho_m^3$$

as a possible candidate to model the evolution of electron clouds, which are undesirable phenomena that appear in accelerators when accelerated charged particles cause disturbances to stray electrons already present in the accelerator. Electron clouds can impede the path of accelerated particles, which reduces the effectiveness of the accelerator [4].

In this representation, ρ_m represents the average electron cloud density at a point after the m-th passage of the bunch. The quadratic coefficient b must be negative to ensure a positive saturation value of electron cloud density and give concavity to the curve (ρ_m, ρ_{m+1}). As the bunch intensity N exceeds a threshold $N_C \approx 7 * 10^{10}$ protons, a becomes greater than 1. For $N > 10^{10}$ protons, the cubic term c, which accounts for perturbations, is positive and about one order of magnitude smaller than a. Fixed points of this map can be interpreted physically as the saturated electron cloud density.

This model and others were tested using computer simulations, and it was found that the cubic map optimally modeled bunch-to-bunch evolution under the parameters of the Relativistic Heavy Ion Collider and the Large Hadron Collider dipoles [4, 2]. Thus, an understanding of the dynamics of this map would be useful in minimizing the electron cloud effect in the operation of accelerators. As the functions studied in this paper are special cases of the function used in this model, albeit over different parameter spaces, understanding the dynamics of these seven cubic maps may help to better characterize the evolution of electron clouds.

Characterization of Fixed Points

For any of the the cubic polynomials $f(x)$ studied here, the nth iterate $f^n(x)$ is a polynomial of degree 3^n, and there can be a maximum of 3^n cycle points of period n (although not necessarily least period n), because the function $f^n(x) - x = 0$ is of degree 3^n and can have no more than 3^n distinct solutions. Therefore, there are three fixed points, not necessarily distinct, of each of the seven functions. (Indistinct fixed points will be treated as the same.)

In this section, we describe the fixed points of each of the seven functions [f-1], $\cdots$, [f-7] listed above and characterize their behavior over different intervals of the parameter c. We do not present proofs for the following statements, but they are simple to verify using techniques found in [3].

Solving the equation $f(x) - x = 0$, [f-]) yields three fixed points:

$$x^{(1.1)} = 0, \quad x^{(1.2)} = \frac{1}{2}(\sqrt{5-4c} - 1) \quad \text{and} \quad x^{(1.3)} = \frac{1}{2}(-\sqrt{5-4c} - 1).$$

The stability of these fixed points is tested through evaluation of $|f'(x)|$ at each fixed point; $|f'| < 1$ implies attracting behavior, while $|f'| > 1$ implies repelling behavior. The point $x^{(1.1)}$ is stable (attracting) on the parameter space $(-1, 1)$ and repels on $(-\infty, -1) \cup (1, \infty)$. At $c = -1$, $x^{(1.1)}$ repels from the left and attracts from the right, whereas at $c = 1$, $x^{(1.1)} = x^{(1.2)}$, and both attract from the left and repel from the right. The point $x^{(1.2)}$ and $x^{(1.3)}$ are both undefined when $c > \frac{5}{4}$. At $c = \frac{5}{4}$, $x^{(1.2)} = x^{(1.3)}$, and both repel from the left and attract from the right. When $c \in (1, \frac{5}{4})$, $x^{(1.2)}$ attracts, while on the interval $(-\infty, 1) \cup (5/4, \infty)$, it repels. The point $x^{(1.3)}$ repels everywhere except at $c = \frac{5}{4}$.

There are two distinct fixed points for [f-2]:

$$x^{(2.1)} = 0 \quad \text{and} \quad x^{(2.2)} = -c.$$

It is worth noting that at $c = 0$, $x^{(2.1)} = x^{(2.2)}$, and here they both repel. At all $c < 0$, $x^{(2.1)}$ repels from the left and attracts from the right; for $c > 0$, this behavior is reversed. The point $x^{(2.2)}$ is repelling for all values of c.

The distinct fixed points of [f-3] are

$$x^{(3.1)} = 0 \quad \text{and} \quad x^{(3.2)} = -\frac{1}{c}.$$

The point $x^{(3.1)}$ attracts from the left and repels from the right for all c. The point $x^{(3.2)}$ is attracting on $(-\infty, -\frac{1}{2}]$ ($c = -\frac{1}{2}$ makes it neutral), and repelling on $(-\frac{1}{2}, 0) \cup (0, \infty)$. At $c = 0$, it is clearly not defined, and the function collapses to a quadratic equation.

Three fixed points exist for [f-4]:

$$x^{(4.1)} = 0, \quad x^{(4.2)} = \sqrt{1-c} \quad \text{and} \quad x^{(4.3)} = -\sqrt{1-c}.$$

The point $x^{(4.1)}$ has behavior similar to $x^{(1.1)}$ with the exception of the values of c that make it non-hyperbolic, $c = \pm 1$. At $c = -1$, $x^{(4.1)}$ becomes an attracting fixed point from both sides. It is clear that $x^{(4.2)}$ and $x^{(4.3)}$ only exist and are distinct for $c < 1$; at $c = 1$, they are equal to each other and to $x^{(4.1)}$, and all three repel. Both $x^{(4.2)}$ and $x^{(4.3)}$ are repelling over the entire domain of c over which they are defined.

Only one distinct fixed point exists for $c \neq 0$ for [f-5], at $x^{(5.1)} = 0$. This fixed point is always neutral, and shows similar behavior to $x^{(2.1)}$ except at $c = 0$. For $c = 0$, the function reduces to the linear equation $y = x$, for which $f(x) = x$, $\forall x \in \mathbb{R}$, so $x^{(5.1)}$ neither repels not attracts.

The function [f-6] has three fixed points:

$$x^{(6.1)} = 0, \quad x^{(6.2)} = \frac{-c + \sqrt{c^2+4}}{2} \quad \text{and} \quad x^{(6.3)} = \frac{-c - \sqrt{c^2+4}}{2}.$$

The point $x^{(6.1)}$ is attracting for all $c \in \mathbb{R}$, while $x^{(6.2)}$ and $x^{(6.3)}$ are repelling over all $c \in \mathbb{R}$.

[f-7] also has three distinct fixed points:

$$x^{(7.1)} = 0, \quad x^{(7.2)} = \frac{-1 + \sqrt{1+4c}}{2c} \quad \text{and} \quad x^{(7.3)} = \frac{-1 - \sqrt{1+4c}}{2c}.$$

The point $x^{(7.1)}$ is an attracting fixed point over all $c \in \mathbb{R}$. The point $x^{(7.2)}$ and $x^{(7.3)}$ only emerge for $c \geq -\frac{1}{4}$ and are undefined at $c = 0$. At $c = -\frac{1}{4}$, $x^{(7.2)} = x^{(7.3)}$, and both are neutral; further testing showed they repel from the left and attract from the right. The point $x^{(7.2)}$ repels on the rest of the parameter space over which it is defined. The point $x^{(7.3)}$ attracts on $(-\frac{1}{4}, -\frac{3}{16}]$, and repels on the rest of the parameter space over which it is defined.

Characterization of 2-Cycles

As finding the 2-cycles of a cubic function f requires finding the roots of the degree nine polynomial $f^2(x) - x = 0$, 2-cycles for all functions studied except the odd functions [f-4] and [f-5] could not be found analytically. (No 2-cycles could be found for [f-2] or [f-6].) As in the previous section, proofs are not shown, but they are not difficult to verify using the provided and derived rules for determining and classifying 2-cycles.

Since for any degree three polynomial f, f^2 is a degree nine polynomial, the equation $f^2(x) - x = 0$ has nine solutions (including complex solutions). Three of these nine are fixed points, leaving six 2-cycle points. For our case, we focus only on the real cycle points.

We evaluated the attracting or repelling behavior of 2-cycles using the chain rule for dynamical systems. In this case, this entails evaluating a cycle denoted $\{a, b\}$ by $|(f^2)'(a)| = |f'(a)f'(b)|$. A cycle is repelling when this quantity is greater than one, and attracting when it is less than one.

Analytic solutions

Function [f-4] has six 2-cycle points, forming three cycles between conjugates. Each pair denoted $\{a, b\}$ below represents a cycle:

$$\left\{x^{(4.4)} = \sqrt{-1-c},\ \ x^{(4.5)} = -\sqrt{-1-c}\right\},$$

$$\left\{x^{(4.6)} = \sqrt{\frac{-c+\sqrt{c^2-4}}{2}},\quad x^{(4.7)} = -\sqrt{\frac{-c+\sqrt{c^2-4}}{2}}\right\},$$

$$\left\{x^{(4.8)} = \sqrt{\frac{-c-\sqrt{c^2-4}}{2}},\quad x^{(4.9)} = -\sqrt{\frac{-c-\sqrt{c^2-4}}{2}}\right\}$$

Points $x^{(4.4)}$ and $x^{(4.5)}$ are both defined at all $c \le -1$. At $c = -1$, they are both equal and the cycle comprising them is neutral. This cycle is again neutral at $c = -2$. On $c \in (-2, -1)$, it attracts, and for all $c \in (-\infty, -2)$, it repels. Both the other cycles are defined for $c \le -2$. The cycle comprising $x^{(4.6)}$ and $x^{(4.7)}$ repels at all c for which it is defined except at $c = -2$, where it is neutral. The cycle formed by $x^{(4.8)}$ and $x^{(4.9)}$ is also neutral for $c = -2$ as well as $c = -\frac{5}{2}$, and is attracting on the interval $(-\frac{5}{2}, -2)$. On the interval $(-\infty, -\frac{5}{2})$, it repels. For [f-5], two distinct solutions were found for the equation $f^2(x) - x = 0$. These solutions are:

$$\left\{x^{(5.2)} = \sqrt{-\frac{2}{c}},\ x^{(5.3)} = -\sqrt{-\frac{2}{c}}\right\}.$$

This 2-cycle is both defined over all negative real numbers and, by the chain rule for dynamical systems, also repelling over their entire parametric domain.

Numerical solutions

For the other functions, the number of 2-cycles was inferred for the number of points of intersection of $f^2(x)$ with the line $y = x$ for a wide range of values of c, then subtracting the number of fixed points of f defined at each c.

In [f-1], two 2-cycle points (forming one complete cycle) emerge for $c < -1$. Four more 2-cycle points emerge below $c \approx -4.6107186$, for a total of six 2-cycle points. Although the behavior of cycle points could also not be inferred analytically for all c over which 2-cycles exists, 2-cycle points corresponding to a wide range of values of c were tested for attracting or repelling behavior using the chain rule for dynamical systems. These tests lead us to believe that two attracting 2-cycle points exist only on $(k, -1)$, where $k \approx -1.451948$.

For [f-3], extensive testing of a range of values of c lead us to conjecture that two 2-cycle points exist for $c \in (-\infty, -\frac{1}{2}]$, and four 2-cycle points exist for $c \in (-\frac{1}{2}, 0)$. Two attracting 2-cycle points could be found only on $(-\frac{1}{2}, k)$, where $k \approx -.403660$. For $c \ge 0$, no 2-cycle points could be found.

[f-7] had behavior similar to [f-4] with regard to the stability of cycle points and their relation to the stability of fixed points. Through extensive testing of c

values, we formed the conjecture that exactly two 2-cycles exist on $(-\frac{1}{4}, -\frac{3}{16})$, and that exactly four exist for all $c \in (-\frac{3}{16}, 0)$. When the behavior of these 2-cycles was tested, two attracting 2-cycles were only found on the interval $(-\frac{3}{16}, k)$, where $k \approx -.173501$; all other 2-cycles were repelling.

Bifurcation Diagrams

Three bifurcation diagrams gave clear indication of the emergence of chaotic behavior. The diagrams that show evidence for chaos are those for [f-1], [f-3], and [f-4], which can be found in Figures 1, 2, and 3, respectively.

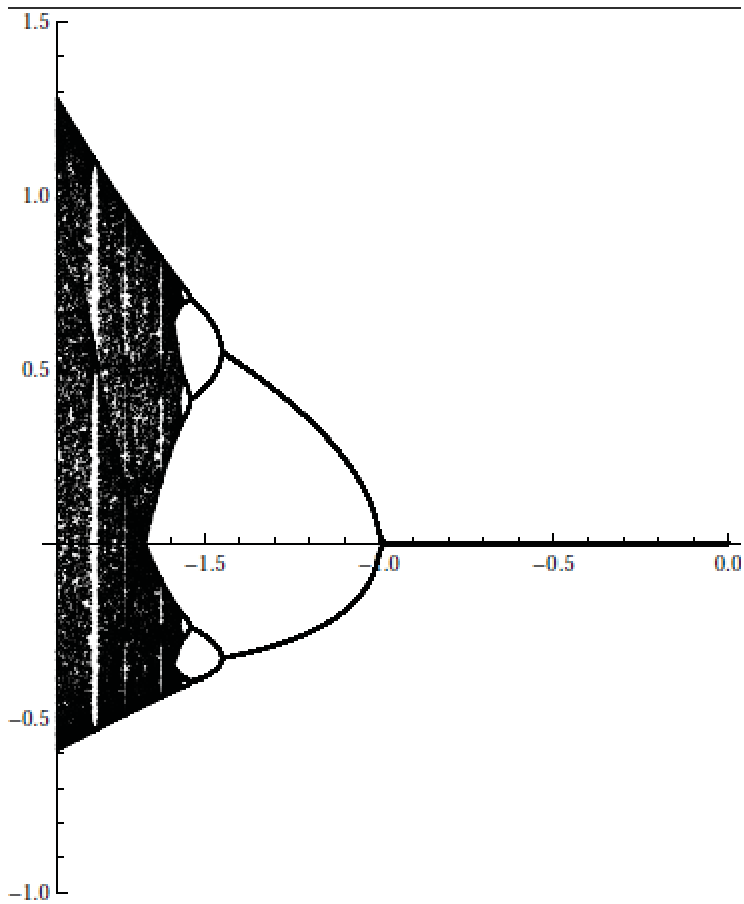


Figure 1: Bifurcation diagram for $f(x) = x^3 + x^2 + cx$, [f-1].

In the bifurcation diagram of [f-1], one stable fixed point exists for $c > -1$, at $x = 0$. This corroborates the fixed-point analysis of the function. Period-doubling occurs at $c < -1$, as predicted through analysis of 2-cycles. At $c \approx -1.58$, chaotic behavior begins. At $c \approx -1.825$, a 3-cycle emerges, which serves as proof of chaotic behavior [5]. Chaotic behavior continues until about $c \approx -1.93$.

The bifurcation diagram of [f-3] showed one stable fixed point at $c < -\frac{1}{2}$, which, as predicted earlier in fixed-point analysis, was $x = -\frac{1}{c}$. Period-doubling begins at $c > -\frac{1}{2}$, and chaotic behavior emerges at $c \approx -0.37$. An island of stability containing a stable 3-cycle can be found at $c \approx -0.34$; period-doubling then occurs again, followed by another descent into chaos. A few other islands of stability can be found throughout. Chaotic behavior ends at $c \approx -.30$.

The bifurcation diagram of [f-4] shows odd period-doubling activity that begins below $c = -1$, when a stable 2-cycle emerges, confirming the 2-cycle analysis. At $c = -2$, it appears that period-doubling should occur again; however, no stable four-cycle develops, as one branch of each bifurcation does not appear in the diagram. However, the branch that does appear in each bifurcation "jumps" to a different value of x at some value of c. Period doubling then continues until chaos begins at about $c = -2.33$. The region of chaos ends at $c \approx -3.0$, where all attracting periodic points disappear.

To understand the nature of the second period-doubling bifurcation of this function, orbit diagrams showing the iterates of $f(x) = x^3 - 2.2x$ were produced with varying initial seeds x. These diagrams plot the iterates of the function against the iteration number. Based on these diagrams, it appears, contrary to both the bifurcation diagram and the results of the 2-cycle analysis, that there are two stable 2-cycles here, as shown in Figures 4 and 5. These 2-cycles correspond to the values of x that would characterize the complete cycles that would have existed had both branches in each of the second bifurcations remained visible.

Generating bifurcation diagrams using different initial seeds caused the "jumps" to occur at different values of c, and could even cause the visible parts of these two two-cycles on the bifurcation diagram to reverse, compared to the diagram shown. It seems likely that the diagram is simply failing to detect all attracting cycles for all values of c, which is known to be possible for maps with multiple critical points.

The bifurcation diagrams of [f-1] and [f-3] both clearly show an island of stability containing a stable 3-cycle,

which implies that these dynamical systems must undergo chaotic behavior [5]. They have periodic orbits of all periods in addition to bounded aperiodic orbits. The same is not true for [f-4]; there is no 3-cycle that is clearly visible, and although it seems based on the qualitative properties of the bifurcation diagram that the system does exhibit chaotic behavior, it is not immediately clear that this is the case. As a result, numerical analysis were conducted in the form of Lyapunov exponents. Table 1 shows the Lyapunov exponents of the function at various values of the parameter c for the initial seed $x = -0.25$. Outside of the range shown in Table 1, the normed summation in the calculation of the Lyapunov exponent diverges.

It is clear from Table 1 that for many values of c in the region that appears to show chaotic behavior in Figure 3, there are positive Lyapunov exponents, implying the presence of chaotic behavior. It is worth noting that Lyapunov exponents for $-2 \leq c \leq -1$ exhibit almost perfect symmetry, in that for $0 < k < 1$,

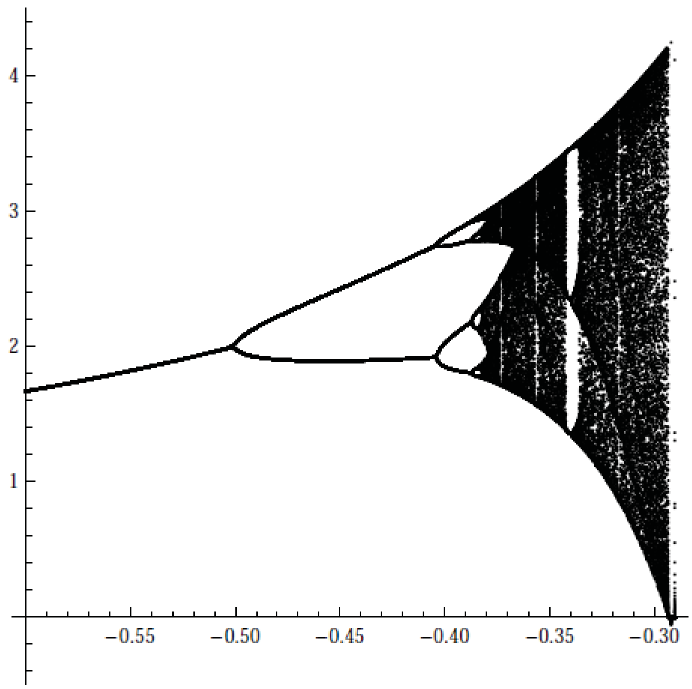


Figure 2: Bifurcation diagram for $f(x) = cx^3 + x^2 + x$, [f-3].

the Lyapunov exponent for $c = -2 + k$ appears to be approximately equal to that of $c = -1 - k$.

One may also notice that the Lyapunov exponent of [f-4] at $c = -2.7$ is negative, even though it appears in the bifurcation diagram that this is in the region of c values in which chaos is exhibited. An orbit diagram plotting the iterates of the seed, shown in Figure 6, was produced, and from this diagram it is clear that we have found the elusive 3-cycle that proves chaotic behavior in this system.

c	$1 < c < 1$	-1	-1.1	-1.2	-1.3	-1.4
λ	< 0	-0.00002	-0.22314	-0.51082	-0.91629	-1.60943
c	-1.5	-1.6	-1.7	-1.8	-1.9	-2.0
λ	-35.3503	-1.60943	-0.91629	-0.51071	-0.22315	-0.00002
c	-2.1	-2.2	-2.3	-2.4	-2.5	-2.6
λ	-0.85739	-0.19283	-0.05882	0.34223	0.49117	0.68189
c	-2.7	-2.8	$-.2.9$	-3.0		
λ	-0.43498	0.80685	0.87815	1.09861		

Table 1: Lyapunov exponents of $f(x) = x^3 + cx$ at $x = -0.25$ far varying c.

The bifurcation diagrams of [f-2] and [f-5] showed no evidence of chaos, as can also be inferred from the observation that neither has a stable 2-cycle. Bifur-

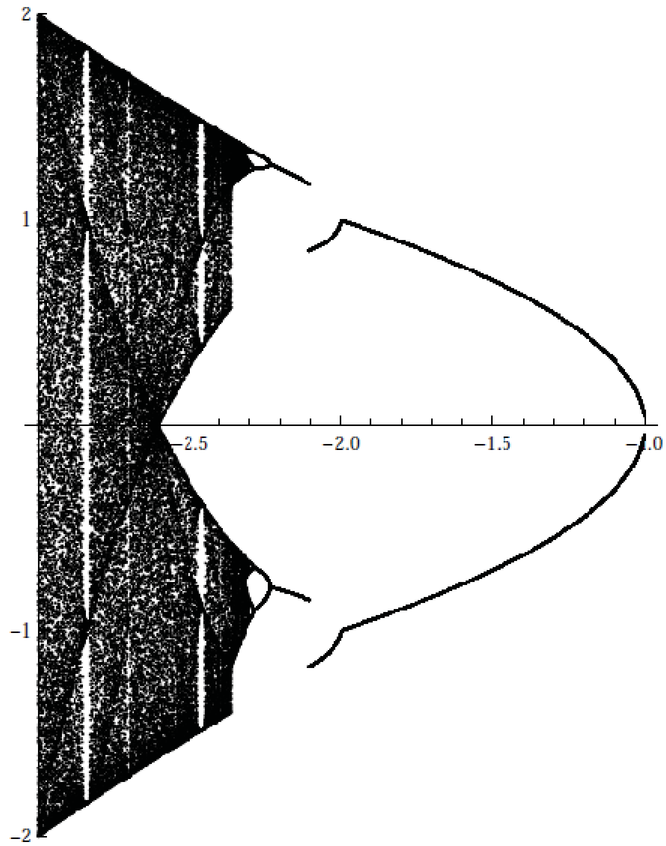


Figure 3: Bifurcation diagram for $f(x) = x^3 + cx$, [f-4].

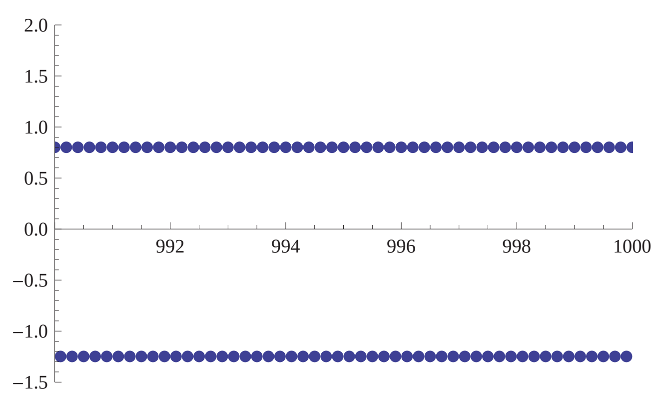


Figure 4: Orbit diagram showing last 100 of 10000 iterates of $f(x) = x^3 - 2.2x$, with a seed of $x_0 = 1$.

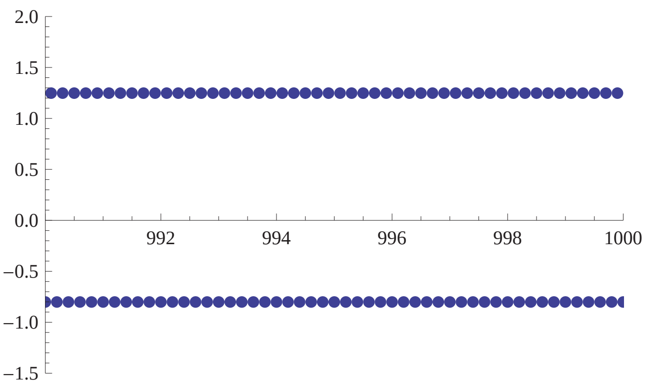


Figure 5: Orbit diagram showing last 100 of 10000 iterates of $f(x) = x^3 - 2.2x$, with a seed of $x_0 = 1.7$.

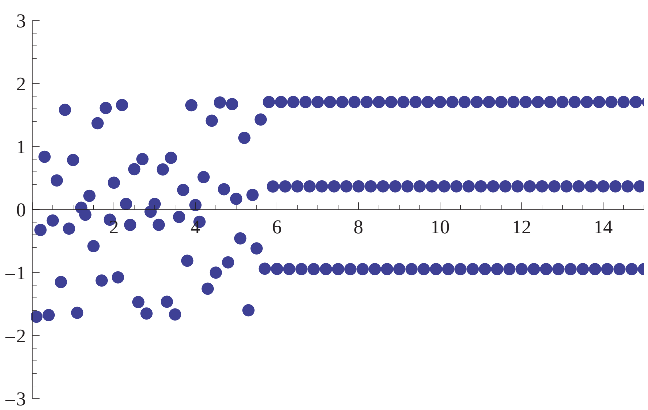


Figure 6: Orbit diagram showing first 150 iterates of $f(x) = x^3 - 2.7x$, with a seed of $x_0 = 1$.

cation diagrams could not be generated for functions [f-6] and [f-7]. As the functions [f-1], [f-3], and [f-4] all exhibit the period-doubling route to chaos over some region of the parameter c, they are qualitatively similar to many maps with one critical point, such as the simple quadratic map and the logistic map. On the other hand, [f-2] and [f-5] clearly showed no chaotic behavior, demonstrating that cubic maps that appear trivially different can have vastly different dynamics. This observation may be of use in characterizing the behavior of cubic maps such as those that may be useful in population genetics and physics. We hope that future research will be conducted to investigate the dynamics of first-order difference equations with two critical points, such as those studied in this paper, to help gain a better understanding of these maps and their relation to the already well-studied quadratic family.

References

[1] J. Bair, and G. Haesbroeck, *Monotonous stability for neutral fixed points.* Bulletin of the Belgian Mathematical Society - Simon Stevin **4** (1997) 639-646.

[2] T. Demma, S. Petracca, F. Ruggiero, G. Rumolo, F. Zimmerman, *Maps for electron cloud density in Large Hadron Collider dipoles.* Physical Review Special Topics - Accelerators and Beams. **10** (2007) 1-5.

[3] R. L. Devaney, An Introduction to Chaotic Dynamical Systems (Second Edition), Menlo Park, Benjamin/Cummings, 1986

[4] U. Iriso, and S. Peggs, *Maps for electron clouds.* Physical Review Special Topics - Accelerators and Beams **8** (2005) 1-8.

[5] T.-Y. Li, and J. A. Yorke, *Period Three Implies Chaos*, American Mathematical Monthly **82** (1975) 985-992.

[6] R. M. May, *Simple mathematical models with very complicated dynamics*, Nature, **261** (1976) 459-467.

[7] R. M. May, *Bifurcations and Dynamic Complexity in Ecological Systems,* Annals of the New York Academy of Sciences, **316** (1979) 517-529.

[8] R. M. May, *Non-Linear Phenomena in Ecology and Epidemiology.* Annals of the New York Academy of Sciences, **357**(1980) 267-281.

[9] F. Mukhamedov, *On the Chaotic Behavior of Cubic* p-*Adic Dynamical Systems*, Mathematical Notes **83** (2008) 428-431.

[10] H. Skjolding, B. Branner-Jorgenson, P. L. Christiansen, H. E. Jensen, *Bifurcations in Discrete Dynamical Systems with Cubic Maps*, SIAM Journal on Applied Mathematics, **43** (1983) 520-534.