Student Projects

Economic Dynamic Modeling: An Overview of Stability

Nathan Berggoetz

Nathan Berggoetz is a senior actuarial science and mathematical economics major. After graduation he plans to work as an actuarial consultant. The content of this paper is the result of an economics independent study with Dr. Lee C. Spector.

Economics often focuses on the end result of changes in the economy. For example, it is important to know now, if increases in government spending this year will affect real income and interest rates in the long run. Economic dynamics seeks to understand the change in economic variables over time and whether or not economic results can be predicted mathematically. In the above example, an understanding of real income and interest rate changes over time would be observed. This analysis allows for better government policy recommendations since recommendations can be tailored to the desired results such as a fast increase in real income and a slow increase in interest rates or vice versa. The focus here will be to consider two dynamic macroeconomic models, the first uses the goods and money market to model the macroeconomy which we will call the IS-LM model, and the second uses the goods and bonds market to model the macroeconomy which we will call the IS-BB model. In this paper we will determine the stability of these two models as a check for their validity. If either model shows instability it must be reformulated. Since the real world does not exhibit unstable markets in which economic variables move toward economic ruin, the correct model for the economy must be stable. If both models exhibit stability then a second check must be performed in order to choose which model is best. The second check involves comparing the time paths for each model and empirical data. It is important to note that these models consider all variables to be in real terms. Money is in nominal terms, which means that it simply serves as a unit of account, and as such will always provide the same service. This implies that to convert money as a measure of something such as income, expenditure, or the supply of money into real terms, the amount of money must be divided by the price level. We consider the price level to be the price of a representative good from goods market, which can be thought of as an average of all prices.

IS-LM Model

To derive the IS-LM model, the goods and money market are used. The goods market represents all possible spending by consumers, businesses, and government. The money market represents money as a commodity desired by individuals just like any other potential product, although we consider it separately from the goods market. A simplified version of Shone's IS-LM model will be used [2]. Consider,

$$e(t) = A + G + Cy(t) - hr(t); \quad A,\, G,\, h > 0,\, 0 < C < 1, \tag{1}$$

where y and r are functions of time t, and $y(t)$, $r(t)$, and $e(t)$ represent real income, interest rates, and real expenditure, respectively. The constants are as follows: A is autonomous expenditure which accounts for any expenditure not already inherent in the model; G is government spending; C, the marginal propensity to consume, is a measure of a given consumer's desire to spend instead of save; h is the coefficient of investment in response to changes in the interest rate r, i.e., a measure of a firm's desire to invest in capital for production. It is known that real income in the goods market $y(t)$ changes proportionally to excess demand, that is,

$$\dot{y} = \frac{dy}{dt} = \alpha[e(t) - y(t)] = \alpha A + \alpha G + (C-1)\alpha y - \alpha h r; \quad \alpha > 0. \tag{2}$$

In the money market, the demand for money M^D is positively related to income and negatively related to interest rates. In this model money supply is assumed exogenous, i.e., determined by factors outside of this model's scope since this model is not concerned with how the Federal Reserve System determines the quantity of money to put into the economy and real money is defined to be $m_0 = \frac{M_0}{p}$, where p is a constant price level and M_0 is the nominal money supply. As noted above, all variables are in real terms. Money is in nominal terms. Therefore, to find the true value of money, the supply of money $M^S(t)$ must be divided by the price level since this will give us a measure of how many actual goods and services the money can buy. For our model we have,

$$M^D(t) = ky - nr; \quad k,\, n > 0, \tag{3}$$

$$M^S(t) = \frac{M_0}{p} = m_0. \tag{4}$$

In the money market, interest rates change proportionally to the excess demand for money, i.e.,

$$\dot{r} = \frac{dr}{dt} = \beta(M^D(t) - m_0) = \beta k y - \beta n r - \beta m_0; \quad \beta > 0. \tag{5}$$

The curve representing the goods market, where $\dot{y} = 0$, is then found to be

$$r = \frac{A + G + (C-1)y}{h}, \tag{6}$$

which we designate as the IS curve. The curve representing the money market, where $\dot{r} = 0$, is then found to be

$$r = \frac{ky - m_0}{n}, \tag{7}$$

which we designate as the LM curve. Note that the IS and LM curves are analogous to simple linear demand and supply curves, respectively. Only in this case, we are concerned with an equilibrium of real income and interest rate instead of an equilibrium of quantity and price. The fixed point of this model, i.e., the economic equilibrium, is a point (y_0, r_0) where $\dot{y} = \dot{r} = 0$. To determine the stability of this fixed point, we will now produce a linear approximation of the differential equations. Equations (2) and (5) above are in the form

$$\frac{dy}{dt} = f(y,\, r) \quad \text{and} \quad \frac{dr}{dt} = g(y,\, r) \quad \text{with} \quad f(y_0,\, r_0) = g(y_0,\, r_0) = 0. \tag{8}$$

Using the substitutions $u = y - y_0$ and $v = r - r_0$, we find that

$$\begin{aligned} \frac{du}{dt} &= \frac{d(y-y_0)}{dt} = \frac{dy}{dt} = f(y,\, r) = f(y_0 + u,\, r_0 + v) \\ &\approx f(y_0,\, r_0) + \frac{\partial f}{\partial y}(y_0, r_0)\, u + \frac{\partial f}{\partial r}(y_0, r_0)\, v, \end{aligned} \tag{9}$$

and, similarly,

$$\frac{dv}{dt} \approx g(y_0,\, r_0) + \frac{\partial g}{\partial y}(y_0, r_0)\, u + \frac{\partial g}{\partial r}(y_0,\, r_0)\, v. \tag{10}$$

Since $f(y_0,\, r_0) = g(y_0,\, r_0) = 0$, the approximations for $\frac{du}{dt}$ and $\frac{dv}{dt}$ can be written in matrix notation:

$$\begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial y}(y_0, r_0) & \frac{\partial f}{\partial r}(y_0, r_0) \\ \frac{\partial g}{\partial y}(y_0, r_0) & \frac{\partial g}{\partial r}(y_0, r_0) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

which gives

$$\begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} (C-1)\alpha & -\alpha h \\ \beta k & -\beta n \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{11}$$

Let

$$A = \begin{bmatrix} (C-1)\alpha & -\alpha h \\ \beta k & -\beta n \end{bmatrix}$$

and let λ_1 and λ_2 denote the possibly complex eigenvalues of the matrix A. Then λ_1 and λ_2 have negative real parts. To see why, consider the determinant, $\det(A)$, and trace, $\text{Tr}(A)$, of the matrix A. We have

$$\lambda_1\lambda_2 = \det(A) = -\beta\alpha n(C-1) + \beta\alpha kh = \beta\alpha[-n(C-1)+kh] \tag{12}$$

and

$$\lambda_1 + \lambda_2 = \text{Tr}(A) = \alpha(C-1) - \beta n. \tag{13}$$

From (1), (2), (3), and (4) we get $\beta\alpha > 0$, $-n(C-1) > 0$, and $kh > 0$. Hence $\det(A) > 0$ implying that $\lambda_1\lambda_2 > 0$. Since $\alpha(C-1)$ and βn are both negative, we see that $\text{Tr}(A) < 0$, implying that $\lambda_1 + \lambda_2 < 0$. In order for $\lambda_1\lambda_2 > 0$ and $\lambda_1 + \lambda_2 < 0$, we must have $\lambda_1 < 0$ and $\lambda_2 < 0$ if λ_1 and $\lambda_2 \in \mathbb{R}$, or the real part of λ_1 is equal to the real part of λ_2 and less than zero.

Thus (y_0, r_0) is an asymptotically stable critical point for the IS-LM model: The general solution to the system of the differential equations (11) is given by

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2; \;\; \lambda_1 \neq \lambda_2 \tag{14}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the possibly complex corresponding eigenvectors of the matrix A. Using this general solution approximation, we see that

$$\lim_{t\to\infty} u(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} v(t) = 0$$

since both λ_1 and λ_2 have negative real parts. If $u(t)$ and $v(t)$ approach zero as t approaches infinity, then $y(t)$ and $r(t)$ approach y_0 and r_0, respectively, given that the initial conditions $y(0)$ and $r(0)$ are close enough to (y_0, r_0).

IS-BB Model

For the IS-BB model, we again use the goods market, but change to the bonds market instead of the money market. This change in markets allows us to analyze the same economy implicit in the IS-LM model with different implications in the time paths of income and interest rates. We are allowed to change markets because of Walras Law, which states that the sum of the excess demands for the goods market, money market, and bonds market must be zero [1]. In essence, if we know the behavior of two of the three markets, the behavior of the third can be deduced. Thus, we can analyze the goods market and bond market and implicitly examine the money market. The bond market is specified as follows:

$$B^d = D_0 + D_r r + D_y y; \;\; D_0 > 0, D_r > 0, D_y > 0, \tag{15}$$

$$B^s = S_0 - S_r r + S_g G; \qquad S_0 > 0, S_r > 0, S_g > 0, \tag{16}$$

where B^d is the demand for bonds, D_0 is exogenous demand which encompasses all things effecting the bond demand not explicitly accounted for within the model, D_r is the constant of proportionality to interest rates r which takes into account the desire to purchase bonds in response to interest rates, and D_y is the constant of proportionality to income y which takes into account the desire to purchase bonds in response to real income. We define B^s as the supply of bonds, where S_0 is the exogenous supply accounting for all factors not explicitly considered in this model, and S_r and S_g are constants of proportionality related to interest rates r and government spending G, respectively. The higher interest rates are, the less willing institutions are to sell bonds because of increased cost. Opposite to this, the more the government spends, the more bonds the government needs to sell to finance its increased expenditure. In the bond market, interest rates r change proportionally to the excess supply of bonds, and it is given as such:

$$\dot{r} = \beta[B^s - B^d] = \beta S_0 - (S_r + D_r)\beta r - \beta D_y y + \beta S_g G - \beta D_0; \;\; \beta > 0. \quad (17)$$

The curve representing the bonds market, where $\dot{r} = 0$, is found to be

$$r = \frac{S_0 + S_g G - D_0 - D_y y}{S_r + D_r}, \quad (18)$$

which we call the BB curve. We continue to use equations (1) and (2) to specify how real income y changes. For our IS-BB model to make economic sense, the slope of the IS curve must be less than the slope of the BB curve because, for the purposes of this paper, this keeps the model from violating Walras Law (for a more technical treatment, see McCaleb and Sellon [1]). We find the condition to be as follows. Let r_{IS} denote the IS curve in (6) and let r_{BB} denote the BB curve given in (18). Then:

$$\frac{dr_{IS}}{dy} < \frac{dr_{BB}}{dy} \quad \Rightarrow \quad \frac{C-1}{h} < \frac{-D_y}{S_r + D_r}. \quad (19)$$

The function $g(y, r)$ now equals $\frac{dr_{BB}}{dt}$. As a result, we change the matrix A accordingly to

$$A = \begin{bmatrix} (C-1)\alpha & -\alpha h \\ -\beta D_y & -(S_r + D_r)\beta \end{bmatrix}$$

and rerun our analysis of eigenvalues to determine the stability of the fixed point (y_0, r_0) of the IS-BB model. To show that λ_1 and λ_2 have negative real parts, consider the determinant and trace of the matrix A:

$$\det(A) = -\alpha\beta(C-1)(S_r+D_r) - \alpha\beta h D_y = -\alpha\beta[(C-1)(S_r+D_r) + hD_y] \quad (20)$$

$$\mathrm{Tr}(A) = \alpha(C-1) - \beta(S_r + D_r) \quad (21)$$

From (2), (17), and (19), we know that $(C-1)(S_r + D_r) + hD_y < 0$ and $-\alpha\beta < 0$, which implies that $\lambda_1\lambda_2 > 0$. From (1), (2), (15), (16), and (17), we know that $\alpha(C-1) < 0$ and $-\beta(S_r + D_r) < 0$, which implies that $\lambda_1 + \lambda_2 < 0$. As in the IS-LM model, the only way for $\lambda_1\lambda_2 > 0$ and $\lambda_1 + \lambda_2 < 0$ to hold is that $\lambda_1 < 0$ and $\lambda_2 < 0$ if λ_1 and $\lambda_2 \in \mathbb{R}$, or that the real part of λ_1 is equal to the real part of λ_2 and less than zero. Thus (y_0, r_0) is an asymptotically stable critical point for the IS-BB model.

Conclusion

The above analysis has shown the stability of the IS-LM and IS-BB models. As a result, a second check must be performed to choose the best model for the economy. The second check involves solving differential equations (2) and (5) for the IS-LM model, and (2) and (17) for the IS-BB model. The parameters of these solutions are then estimated from empirical data and checked for statistical significance. Hopefully, the empirical data will clearly show which model better represents reality.

The differences in the time paths for interest rates r and real income y in the two models are noteworthy. The IS-LM model suggests that the government, desiring to increase spending, will spend money before it has obtained the funds to do so. The IS-BB model suggests that the government will obtain financing for its increased spending through financial markets before it spends the money. These two scenarios, yielding very different results in their effect on the economy, require further investigation [3].

References

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