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Exploring Properties of Cayley Graphs of the Integers with Infinite Generating Sets

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Abstract The work in this paper was motivated by the question that Richard E. Schwartz posed in 2008: "Are the metric spaces $(\mathbb{Z}, d_2)$ and $(\mathbb{Z}, d_3)$, where d_g is the word metric associated with the infinite generating set $\{\pm g^n | n = 0, 1, 2, \dots\}$, quasi-isometric?"

In this paper, we recover several known results with novel methods, and derive new results. First, we show that the associated Cayley graphs, $C_2 = Cay(\mathbb{Z}, \{\pm 2^n\})$ and $C_3 = Cay(\mathbb{Z}, \{\pm 3^n\})$, of these metric spaces are not isometric. Then the bi-Lipschitz equivalence between them is considered. Next, a few properties (hyperbolicity, metric ends, and asymptotic dimension) are discussed and it is demonstrated why these properties cannot be used to answer Schwartz's question. Finally, the main results prove that particular types of maps (among them polynomial maps with rational coefficients) are not quasi-isometries between C_2 and C_3.

Introduction

One of the unsolved problems in geometric group theory states:

[13, Problem 2] *Let a and b be integers greater than 1, and let d_a and d_b be the metrics on $\mathbb{Z}$ associated with the generating sets $A = \{a^i\}_{i=0}^{\infty}$ and $B = \{b^j\}_{j=0}^{\infty}$, respectively. Are the metric spaces $(\mathbb{Z}, d_a)$ and $(\mathbb{Z}, d_b)$ quasi-isometric?*

Richard E. Schwartz first posed a particular case of this problem in 2008 [14, Problem 6]: *Are the metric spaces $(\mathbb{Z}, d_2)$ and $(\mathbb{Z}, d_3)$ quasi-isometric?* This question is still open and motivated the work in this paper.

Let G be a group with identity e, and let S be a (finite or infinite) set of generators for G. By convention [5, p. 78], we will assume that S does not contain e and is *symmetric*, i.e., $s \in S$ if and only if $s^{-1} \in S$. A *word with respect to* S is a finite sequence of elements from S (possibly with repetition). The *word length* of an element $g \in G$ ($g \neq e$) with respect to S, $l_S(g)$, is the smallest positive integer n such that there exists a sequence $(s_1, s_2, \dots, s_n)$ of elements of S, so that $g = s_1 s_2 \cdots s_n$. Define $l_S(e) = 0$. The function $l_S : G \to \mathbb{N}_0$ defined above is called a *length function*.

The length function induces a metric $d_S : G \times G \to \mathbb{N}_0$ on G, called the *word metric*, defined by

$$d_S(x, y) = l_S(x^{-1}y).$$

The word metric on G depends on the generating set S. However, the word metric on finitely generated groups is unique up to quasi-isometry (defined below) [5, Proposition IV.22(i)]. Since d_S obtains just integer values, (G, d_S) is a discrete metric space. Hence, a group G is viewed as a geometric object by looking at the corresponding Cayley graph of G. The *Cayley graph* of G, $Cay(G, S)$, is the graph whose vertex set is G and the set of edges consists of $(g_1, g_2) \in S \times S$ such that $d_S(g_1, g_2) = 1$. $Cay(G, S)$ can be made into a metric space by first making each edge a metric space isometric (by the natural identification) to the segment $[0, 1]$, then defining the length of a path between two vertices in a natural way, and finally defining the distance between two points to be the minimum of the lengths of all paths between these two points of the graph. Now the word metric on G is defined using its corresponding Cayley graph: for given $g_1, g_2 \in G$, $d_S(g_1, g_2)$ is the length of the shortest path in the Cayley graph between the vertices represented by g_1, g_2, i.e., the number of edges of the shortest path between these vertices. This minimal-length edge path joining g_1 and g_2 is called a *geodesic path*. Thus, $Cay(G, S)$ is a geodesic metric space [5, Example IV.18(ii)]. Please refer to P. de la Harpe [5] and J. Meier [12] for more details.

In this paper, the additive group $\mathbb{Z}$ with the infinite generating sets $\{\pm g^n \mid n \in \mathbb{N}_0\}$, $g \in \mathbb{Z}^+$, (in particular, for $g = 2, 3$) is discussed. The corresponding word metric d_g on each of these spaces is described by Melvyn B. Nathanson in [14]. Nathanson showed that every integer n has a unique representation (called *special g-adic representation of n*) in the form $n = \sum_{i=0}^{\infty} \varepsilon_i g^i$, such that:

1. [14, Theorem 3] if g is even, then
   (a) $\varepsilon_i \in \{0, \pm 1, \pm 2, \dots, \pm g/2\}$ for all $i \in \mathbb{N}_0$,
   (b) $\varepsilon_i \neq 0$ for only finitely many nonnegative integers i,
   (c) if $|\varepsilon_i| = g/2$, then $|\varepsilon_{i+1}| < g/2$ and $\varepsilon_i \varepsilon_{i+1} \geq 0$;
2. [14, Theorem 6] if g is odd ($g \geq 3$), then
   (a) $\varepsilon_i \in \{0, \pm 1, \pm 2, \dots, \pm (g-1)/2\}$ for all $i \in \mathbb{N}_0$,
   (b) $\varepsilon_i \neq 0$ for only finitely many nonnegative integers i.

In addition, Nathanson proved that $d_g(n,0) = l_g(n) = \sum_{i=0}^{\infty} |\varepsilon_i|$ in the metric space $(\mathbb{Z}, d_g)$, where d_g is the word metric associated with the generating set $\{\pm g^i \mid i \in \mathbb{N}_0\}$.[§] We will call this distance formula *Nathanson's length formula*. Specifically, Nathanson showed that every integer has a unique 2-adic representation (of shortest length) as a finite sum and differences of distinct powers of 2 in which no two consecutive powers of 2 occur [14, Theorem 4]. In addition, he proved that every integer has a unique 3-adic representation (of shortest length) as a finite sum and differences of distinct powers of 3 [14, Theorem 7].

Next we describe several levels of similarities between metric spaces [11]: Let (X, d_X) and (Y, d_Y) be given metric spaces.

A map $f : X \to Y$ is an *isometry* if it is onto and an *isometric embedding*, i.e.,

$$d_Y(f(x), f(x')) = d_X(x, x') \text{ for all } x, x' \in X.$$

The metric spaces X and Y are *isometric* if there exists an isometry between them.

Isometry is the strongest type of similarity between metric spaces that preserves the local information of the spaces. The more general notion is the bi-Lipshitz equivalence.

The map $f : X \longrightarrow Y$ is a *bi-Lipschitz embedding* if there is a constant $k \geq 1$ such that for all $x, x' \in X$,

$$\frac{1}{k}\, d_X(x, x') \leq d_Y(f(x), f(x')) \leq k\, d_X(x, x').$$

If, in addition, f is a bijection, then the map f is called a *bi-Lipschitz equivalence*. The metric spaces X and Y are *bi-Lipschitz equivalent* if there exists a bi-Lipschitz equivalence between them.

Note that bi-Lipschitz equivalence also preserves local information of the spaces.

The concept of quasi-isometry we are interested in compares metric spaces based on the large scale shape (coarse structure) of the spaces (without preserving the local information of the spaces in question).

A map $f : X \longrightarrow Y$ is a *quasi-isometry* if it is a *quasi-isometric embedding*, i.e., there are constants $k \geq 1, c \geq 0$ such that for all $x, x' \in X$,

$$\frac{1}{k} d_X(x, x') - c \leq d_Y(f(x), f(x')) \leq k d_X(x, x') + c,$$

and it has a *quasi-dense image*, i.e., there is a constant $L \in \mathbb{R}^+$ such that

$$(\forall y \in Y)\ (\exists x \in X)\ d_Y(f(x), y) \leq L.$$

The metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry between them.

Note that for uniformly discrete metric spaces, a quasi-isometry is a bi-Lipschitz equivalence if and only if it is a bijection.

As noted above, the notion of distance on a group is independent of a given finite generating set up to quasi-isometry, i.e., if G is a group and S and S' are two finite generating sets, then (G, d_S) and $(G, d_{S'})$ are quasi-isometric. Note that in this case, the corresponding Cayley graphs are locally finite, i.e., every vertex has a finite number of edges to which it is incident.

[§]Note that the generating sets in Nathanson's paper include the additive identity 0, but that does not affect Nathanson's results used in this paper (namely, the zero was not used in computing the distance with Nathanson's length formula).

But the generating sets of the additive group of integers discussed in this paper are infinite generating sets, which means this result does not apply to them. Moreover, the corresponding Cayley graphs are not locally finite graphs.

For these two spaces, $C_2 = Cay(\mathbb{Z}, \{\pm 2^n\})$ and $C_3 = Cay(\mathbb{Z}, \{\pm 3^n\})$, it was already known that they are not isometric, that neither one is hyperbolic, that both are one ended [10], that both have infinite asymptotic dimension, and that the identity map is not a quasi-isometry of the graphs [3, 13]. In the next section, we give a new proof that $C_2 = Cay(\mathbb{Z}, \{\pm 2^n\})$ and $C_3 = Cay(\mathbb{Z}, \{\pm 3^n\})$ are not isometric using Nathanson's length formula. In addition, what is known and unknown about these two spaces in terms of being bi-Lipschitcz equivalent is discussed. In the following section, properties such as hyperbolicity, metric ends, and the asymptotic dimension are considered. We give novel proofs that both metric spaces are not hyperbolic and have infinite asymptotic dimension, and in addition, we survey what is known about C_2 and C_3 in terms of the other property. As a result of these discussions, these properties cannot be used to distinguish between the two metric spaces C_2 and C_3 in terms of quasi-isometry.

Our final section contains the main results. We consider two kinds of maps that are natural candidates for a quasi-isometry between $(\mathbb{Z}, d_2)$ and $(\mathbb{Z}, d_3)$. The first is a map in terms of the coefficients in the special 2- and 3-adic representations, and the second is an arbitrary polynomial with rational coefficients. We prove that neither can provide the needed quasi-isometry, because both fail to be quasi-isometric embeddings.

C_2 and C_3 are not isometric

In this section, we first explore these two spaces by looking at their properties more locally. Namely, before we look at the more general notion of quasi-isometry, we start our investigation of these spaces by proving that they are not isometric.

Remark 1. *Note that by definition, isometry implies quasi-isometry.*

Remark 2. *The metric d_3 is translation invariant, since by definition*

$$d_3(a+c, b+c) = l_3(a+c-(b+c)) = l_3(a-b) = d_3(a,b).$$

Thus, given a map $f : (\mathbb{Z}, d_2) \to (\mathbb{Z}, d_3)$, for any map $g : (\mathbb{Z}, d_2) \to (\mathbb{Z}, d_3)$ defined with $g(x) = f(x) + M$, $M \in \mathbb{Z}$, it follows that $d_3(f(x), f(y)) = d_3(g(x), g(y))$, for any $x, y \in \mathbb{Z}$. Hence, f is a (quasi-)isometry if and only if g is a (quasi-)isometry.

Theorem 3. *The spaces C_2 and C_3 are not isometric.*

Proof. ¶ Suppose there exists an isometry $f : (\mathbb{Z}, d_2) \to (\mathbb{Z}, d_3)$. Without loss of generality by Remark 2, assume that $f(0) = 0$. Let $k \in \mathbb{N}_0$. Then

$$d_2(2^{k+1}, 0) = d_3(f(2^{k+1}), 0) = 1 \text{ and } d_2(2^k, 0) = d_3(f(2^k), 0) = 1$$

Hence, $f(2^{k+1}) = \pm 3^s$ and $f(2^k) = \pm 3^r$, for some $r, s \in \mathbb{N}_0$. Since f is injective, $f(2^{k+1}) \neq f(2^k)$.

¶M. Duchin alerted us that it could be seen why C_2 and C_3 are not isometric, by looking at particular triangles: C_2 has triangles such as (0, 1, 2), but C_3 does not have any such triangles (since the sum of two powers of three can't be a power of three). This can be fully demonstrated by the interested reader. The proof presented here is a new proof as an application of Nathanson's length formula.

Consider the following two cases:

Case 1: Let $r = s$. Then, $f(2^{k+1}) = -f(2^k)$. Therefore $|f(2^{k+1}) - f(2^k)| = 2 \cdot 3^s = -3^s + 3^{s+1}$. Hence, by [14, Theorem 7] $d_3(f(2^{k+1}), f(2^k)) = 2$.

Case 2: Let $r \neq s$. Then $f(2^{k+1}) - f(2^k) = \pm 3^r \pm 3^s$. Thus, by [14, Theorem 7] $d_3(f(2^{k+1}), f(2^k)) = 2$.

In either case, a contradiction is reached since by definition of isometry

$$d_3(f(2^{k+1}), f(2^k)) = d_2(2^{k+1}, 2^k) = 1.$$

Hence, f cannot be an isometry between $(\mathbb{Z}, d_2)$ and $(\mathbb{Z}, d_3)$. □

By Remark 1 and Theorem 3 it follows that C_2 and C_3 are not isometric, but they may still be quasi-isometric.

Next, we discuss the more general notion of isometry: the bi-Lipschitz equivalence.

As was mentioned before, for some discrete metric spaces (such as the ones discussed in this paper), bi-Lipschitz equivalence is the same as quasi-isometry.

Nathanson proved that the identity map $id : (\mathbb{Z}, d_a) \to (\mathbb{Z}, d_b)$ (where $a, b \in \mathbb{Z}$, $a, b > 1$ and d_a and d_b are the metrics associated with the generating sets $\left\{\pm a^i\right\}_{i=0}^{\infty}$ and $\left\{\pm b^i\right\}_{i=0}^{\infty}$ respectively) is a bi-Lipschitz equivalence if and only if $a^n = b^m$, for some $m, n \in \mathbb{Z}^+$ [13, Theorems 3.1, 3.2]. Hence, the identity map between C_2 and C_3 is not a bi-Lipschitz equivalence, i.e., quasi-isometry, since $2^n \neq 3^m$, for $n, m \in \mathbb{Z}^+$. However, it is still an open question whether C_2 and C_3 are bi-Lipschitz equivalent (i.e., quasi-isometric).

The natural question to ask next was: Can a quasi-isometric invariant be found that can be used to distinguish between C_2 and C_3? This is the subject of the next section.

Properties of C_2 and C_3

This section discusses a few properties that are known to be quasi-isometric invariants (properties of metric spaces that are preserved under quasi-isometry).

The first property discussed is the hyperbolicity (which is a quasi-isometric invariant for all geodesic metric spaces [1, Corollary 2.26]).

Gromov [7] introduced hyperbolicity of a metric space in the late 1980s. The definition of hyperbolicity used here is equivalent to the one given in [7] (see page 2 of [4]).

Let $\Gamma = \{V(\Gamma), E(\Gamma)\}$ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For any $a, b \in V(\Gamma)$, let $d(a, b)$ denote the distance given by the minimal-length edge path from a to b.

The metric space Γ is δ-hyperbolic for some non-negative real number δ if it satisfies Gromov's four point condition [2].

Let $a, b, c, d \in V(\Gamma)$. Define M_1, M_2, and M_3 as

$$\begin{aligned} M_1 &= d(a,b) + d(c,d) \\ M_2 &= d(a,c) + d(b,d) \\ M_3 &= d(a,d) + d(b,c). \end{aligned}$$

Let M and N be the two largest values among M_1, M_2, M_3 with $M \geq N$.
Define

$$hyp(a,b,c,d) = M - N.$$

Note that $hyp(a,b,c,d) = 0$ whenever two elements among a,b,c,d are equal. The graph Γ is *δ-hyperbolic* for $\delta \geq 0$ if

$$\delta \geq \frac{1}{2} hyp(a,b,c,d), \quad \text{for any } a,b,c,d \in V(\Gamma),$$

$$\text{i.e., } \delta \geq \frac{1}{2} max_{a,b,c,d \in V(\Gamma)} \{hyp(a,b,c,d)\}.$$

Γ is *hyperbolic* if it is a δ-hyperbolic for some $\delta \geq 0$.

Theorem 4. *The Cayley graph $C_g = Cay(\mathbb{Z}, d_g)$ (where $g \in \mathbb{N}$ and $g \geq 2$) is not hyperbolic.*

Proof. Assume that C_g is δ-hyperbolic. Take $n \in \mathbb{N}$ such that $n > \delta$. Let

$$a = \sum_{i=0}^{2n+1} g^{2i}, \qquad b = \sum_{i=n+1}^{2n+1} g^{2i}, \qquad c = \sum_{i=0}^{n} g^{2i}, \quad and \quad d = 0.$$

By Nathanson's length formula we have

$$\begin{aligned} & d_g(a,b) = d_g(a,c) = d_g(b,d) = d_g(c,d) = n+1, \\ \text{and} \quad & d_g(a,d) = d_g(b,c) = 2n+2. \end{aligned}$$

Thus,

$$\begin{aligned} M_1 &= d_g(a,b) + d_g(c,d) = 2n+2 \\ M_2 &= d_g(a,c) + d_g(b,d) = 2n+2 \\ M_3 &= d_g(a,d) + d_g(b,c) = 4n+4. \end{aligned}$$

Then $M = M_3$ and $N = M_1 = M_2$. Hence,

$$\begin{aligned} \frac{1}{2} hyp(a,b,c,d) = \frac{1}{2}(M - N) &= \frac{1}{2}[4n + 4 - (2n+2)] \\ &= n + 1 \\ &> \delta. \end{aligned}$$

This contradicts the assumption and proves the theorem. □

Thus, by Theorem 4 it follows that neither graph C_2 nor C_3 is hyperbolic.

Another property discussed involves the metric ends (which is also a quasi-isometric invariant [9, Theorem 6]).

Let Γ be a graph. A *ray* in Γ is a sequence $(x_0, x_1, \ldots)$ of distinct vertices x_i of Γ, such that x_i and x_{i+1} are adjacent for $i \geq 0$. A set of vertices F *separates* vertices x and y in Γ if every path from x to y contains a vertex of F. The set F *separates* sets of vertices A and B if it separates any vertex in A from any vertex in B.

A ray is *metrically transient* if every infinite subset of vertices has an infinite diameter. Two metrically transient rays are equivalent if they cannot be separated by a bounded set of vertices. The *metric ends* of a graph are equivalence classes of metrically transient rays. (For more details on metric ends refer to [10]).

Krön showed that $C_g, g \in \mathbb{N}, g > 1$ has one metric end [10, Example 3.16]. Therefore, C_2 and C_3 have one metric end, thus sharing the same quasi-isometric invariant.

The third property that we discuss in this section is the asymptotic dimension of a metric space X, defined by Gromov [6]. The asymptotic dimension is a coarse invariant [15], i.e., this property is preserved under coarse equivalence (a more general equivalence than the quasi-isometry).

A metric space X has *asymptotic dimension* $\leq n$ if, for every $d > 0$, there is an $R > 0$ and $n+1$ d-disjoint, R-bounded families $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n$ of subsets of X such that $\bigcup_{i=0}^{n} \mathcal{U}_i$ is a cover of X.

A family $\mathcal{U}$ of subsets of X is R-bounded if $\sup\{\operatorname{diam} U \mid U \in \mathcal{U}\} \leq R$.

Also, $\mathcal{U}$ is said to be d-disjoint if $d(x,y) > d$ whenever $x \in U, y \in V$, $U \in \mathcal{U}, V \in \mathcal{U}$, and $U \neq V$.

It is known that metric spaces containing an isometrically embedded copy of $\mathbb{Z}^n$ for every n cannot have a finite asymptotic dimension [15, Remark 9.20]. We use this fact to provide a novel proof that the asymptotic dimension of C_2 and C_3 is infinite.

Theorem 5. *C_g, $g = 2,3$, contains an isometrically embedded copy of $\mathbb{Z}^n$ for every n.*

Proof. [‖] Consider C_2 and the set $S = \{2^{2k} \mid k = 0,1,2,\ldots\}$. Partition S into n disjoint subsets of form

$$S_i = \{\ldots, s^i_{-2}, s^i_{-1}, s^i_1, s^i_2, \ldots\},$$

where for every $j \in \mathbb{Z}$, $s^i_j \in S$. Define a map $f : \mathbb{Z}^n \to C_2$ as follows:

$$f(m_1, m_2, \ldots, m_n) = \sum_{\substack{i \\ m_i > 0}} (s^i_1 + s^i_2 + \cdots + s^i_{m_i}) + \sum_{\substack{j \\ m_j < 0}} (s^j_{-1} + s^j_{-2} + \cdots + s^j_{m_j}).$$

The 1-1 property of this map follows from the definition of the set S and the way it was partitioned, as well as the uniqueness of the special 2-adic representations of the integers. It can easily be seen that the map is distance preserving:

$$d((0,0,\ldots,0),(m_1,m_2,\ldots,m_n)) = \sum_{i=1}^{n} |m_i| = d_2(0, f(m_1,m_2,\ldots,m_n)),$$

where d is the usual word metric on $\mathbb{Z}^n$ with the usual set of generators. Hence, $\mathbb{Z}^n$ is isometrically embedded in C_2.

Similarly, it can be shown that C_3 contains an isometrically embedded copy of $\mathbb{Z}^n$ for every n, by partitioning the set $S = \{3^k \mid k = 0,1,2,\ldots\}$. □

Hence, C_2 and C_3 here have the same asymptotic dimension as well.

The discussion in this section shows that the most common quasi-isometry invariants fail to distinguish between C_2 and C_3.

Maps that fail to be quasi-isometries between C_2 and C_3

This section contains the main results of this paper. Namely, a few maps between the spaces C_2 and C_3 will be considered. It will be proved that they are not quasi-isometries.

Note that Nathanson [13] proved that the identity map between C_2 and C_3 is not a quasi-isometry. Here, other maps will be considered and it will be shown they fail to be quasi-isometries since they fail to be quasi-isometric embeddings.

Theorem 6. *Let $f : (\mathbb{Z}, d_2) \to (\mathbb{Z}, d_3)$ be a map defined with*

$$f(a) = \sum_{i=0}^{\infty} \varepsilon_i 3^i,$$

‖The idea used here was provided by Z. Šunić.

where $a = \sum_{i=0}^{\infty} \varepsilon_i 2^i \in \mathbb{Z}$*, satisfying the conditions of [14, Theorem 3].*

Then f *is not a quasi-isometry.*

Proof. Assume that f, as defined above, is a quasi-isometry.
Then there exist constants $k \geq 1$ and $c \geq 0$, such that

$$\frac{1}{k} d_2(a,b) - c \leq d_3(f(a), f(b)) \leq k d_2(a,b) + c.$$

Choose $n \in \mathbb{N}$ such that $n > 2k + c$. Then take $a = 1 + 2 + 2^2 + \cdots + 2^{n-1}$ and $b = 0$. By Nathanson's length formula for C_2, $d_2(a, 0) = l_2(2^n - 1) = 2$. In addition, by Nathanson's length formula for C_3,

$$d_3(f(a), f(0)) = l_3(1 + 3 + 3^2 + \cdots + 3^{n-1}) = n.$$

Thus,

$$n = d_3(f(a), f(b)) \leq k d_2(a,b) + c = 2k + c,$$

which contradicts the way n was chosen ($n > 2k + c$). Thus, f is not a quasi-isometry. □

Next, we adopt the idea of Duchin and White [3] to use the fact that 2 is a primitive root mod 3^n to compare d_2 and d_3.

Lemma 7. *[8, Lemma 3.5] Let* $f(x)$ *be a polynomial in* $\mathbb{Z}[x]$*. If* $a, b \in \mathbb{Z}$ *such that* $a \equiv b \bmod m$ *for some* $m \in \mathbb{N}$*, then* $f(a) \equiv f(b) \bmod m$*.*

Recall that U_n is the multiplicative group of units of $\mathbb{Z}_n$.

Theorem 8. *[8, Theorem 6.7 (Part c of the proof)] Let* $q \in \mathbb{N}$ *be an odd prime. If* g *is a generator of the group* U_{q^2}*, then* g *is a generator of the group* U_{q^e} *for all* $e \geq 2$*.*

Corollary 9. *The element 2 is a generator of the group* U_{3^e} *for all* $e \geq 2$*.*

Proof. It follows immediately from Theorem 8, since 2 is a generator of U_9. □

Lemma 10. *Let* $f(x) \in \mathbb{Z}[x]$*. If the set* $A = \{d_3(f(a), 0) \mid a \in \mathbb{N}\}$ *is unbounded above, then* f *fails the upper bound of the quasi-isometric embedding condition. Namely, there do not exist constants* $k \geq 1$ *and* $c \geq 0$ *such that,*

$$d_3(f(a), f(b)) \leq k d_2(a,b) + c,$$

for all $a, b \in \mathbb{Z}$*.*

Proof. Let $k \geq 1$ and $c \geq 0$ be given. Since the metric d_3 is translation invariant by Remark 2, take $f(0) = 0$.

Since A is unbounded above, we can choose $a \in \mathbb{N}$ such that $d_3(f(a), 0) > 3k + c$. Take $m \in \mathbb{N}$ so that $3^m > \max\{a, |f(a)|\}$. There are two cases for a:

Case 1. $a \in U_{3^m}$. By Corollary 9, choose $l \in \mathbb{N}$ such that $2^l \equiv a \mod 3^m$. Then by Lemma 7,

$$f(2^l) \equiv f(a) \mod 3^m. \tag{1}$$

Note that the condition $3^m > |f(a)|$ implies that in Nathanson's special representation of $f(a)$ the largest power of 3 has an exponent that is less or equal to m. The equation (1) implies that $f(2^l) = f(a) + n3^m$, for some $n \in \mathbb{Z}$. Using Nathanson's algorithm [14, p. 2011] for obtaining the special 3-adic representation of $f(2^l)$, we see that adding $n3^m$ to the special 3-adic representation of $f(a)$ adds/subtracts powers of form 3^j, for $j \geq m$. Hence, the addition of $n3^m$ can remove only the 3^m term from the representation of $f(a)$ and

$$\begin{aligned} d_3(f(2^l), 0) &\geq d_3(f(a), 0) - 1 \\ &> 3k + c - 1 \\ &\geq k + c \\ &= kd_2(2^l, 0) + c. \end{aligned}$$

Hence, the upper bound of the quasi-embedding condition fails.

Case 2. $a \notin U_{3^m}$. Then $3|a$, i.e., $a = 3s$ for some $s \in \mathbb{Z}^+$, hence $(a-1)$ is an element of U_{3^m}. Therefore, we can choose $j \in \mathbb{N}$ such that $2^j \equiv (a-1) \bmod 3^m$. By arguments similar to those in Case 1, we obtain

$$\begin{aligned} d_3(f(2^j + 1), 0) &\geq d_3(f(a), 0) - 1 \\ &> 3k + c - 1 \\ &\geq 2k + c \\ &= kd_2(2^j + 1, 0) + c. \end{aligned}$$

Again, the upper bound of the quasi-embedding condition fails.

In either case, we found integers for which the upper bound of the quasi-embedding condition for the constants k and c fails. □

Lemma 11. *Let $a, b \in \mathbb{Z}$. Then for any positive integer m, $d_3(ma, mb) \leq md_3(a, b)$.*

Proof. Since the metric d_3 is translation invariant by Remark 2, take $b = 0$.

Let $a \in \mathbb{Z}$ have the special 3-adic representation,

$$a = \varepsilon_0 + \varepsilon_1 3 + \cdots + \varepsilon_n 3^n,$$

where $\varepsilon_i \in \{0, \pm 1\}$ and n is some nonnegative integer. Let

$$\sum_{i=0}^{n} |\varepsilon_i| = k.$$

Hence $d_3(a, 0) = k$. Then,

$$\begin{aligned} ma = &= m(\varepsilon_0 + \varepsilon_1 3 + \cdots + \varepsilon_n 3^n) \\ &= m\varepsilon_0 + m\varepsilon_1 3 + \cdots + m\varepsilon_n 3^n. \end{aligned} \tag{2}$$

Notice that equation (2) shows one representation of ma using

$$\sum_{i=0}^{n} m|\varepsilon_i| = m \sum_{i=0}^{n} |\varepsilon_i| = mk$$

elements of the generating set S_3. Thus, $d_3(ma, 0) \leq mk = md_3(a, 0)$. □

Lemma 12. *Let $f : \mathbb{Z} \to \mathbb{Z}$ be a nonconstant polynomial with integer coefficients. Then the set $A = \{d_3(f(a), 0) \mid a \in \mathbb{N}\}$ is unbounded above.*

Proof. Assume the set A is bounded above and $m = \max A$. Then there exists $a \in \mathbb{N}$, such that $d_3(f(a), 0) = m$, i.e.,

$$f(a) = \varepsilon_1 3^{e_1} + \cdots + \varepsilon_m 3^{e_m},$$

where $\varepsilon_i \in \{\pm 1\}$. Let k be the exponent of the largest power of 3 that appears in the special 3-adic representation of a.

Consider the real polynomial function $g : \mathbb{R} \to \mathbb{R}$, such that $g|_{\mathbb{Z}} = f$.

First, suppose $\lim_{x\to\infty} g(x) = \infty$.

Consider the set $B = \{a + 3^{k+e_m+i} \mid i \in \mathbb{N}\}$. Note that $b \equiv a \mod 3^{k+e_m}$, for all $b \in B$, hence Lemma 7 implies that $f(b) \equiv f(a) \mod 3^{k+e_m}$. Since B is unbounded above and g is strictly monotone increasing for large x, choose $b \in B$ such that $f(b) > f(a)$. Therefore, $f(b)$ must have the form

$$f(b) = \varepsilon_1 3^{e_1} + \cdots + \varepsilon_m 3^{e_m} + \varepsilon_{m+1} 3^{e_{m+1}} + \varepsilon_{m+2} 3^{e_{m+2}} + \cdots + \varepsilon_M 3^{e_M},$$

for some $M > m$ and $\epsilon_i \in \{0, \pm 1\}$ for $i > m$. Note that not all of ϵ_i, $i > m$ are zeros. But then $d_3(f(b), 0) > m$, which is a contradiction.

If $\lim_{x\to\infty} g(x) = -\infty$, in a similar way, we can find $b \in B$ such that $d_3(f(b), 0) > d_3(f(a), 0) = m$, which is a contradiction.

Thus, the set A is unbounded above. □

Theorem 13. *Let $f : \mathbb{Z} \to \mathbb{Z}$ be a nonconstant polynomial with rational coefficients. Then f fails to be a quasi-isometry from C_2 to C_3.*

Proof. Suppose $f(x) \in \mathbb{Q}[x]$ is a quasi-isometry from C_2 to C_3. Then $f(x)$ has the form

$$f(x) = \frac{c_m}{d_m} x^m + \ldots + \frac{c_1}{d_1} x + \frac{c_0}{d_0}, \quad m \in \mathbb{N},\ c_i, d_i \in \mathbb{Z}, \quad i = 0, \ldots, m,$$

where c_i and d_i are relatively prime.

Since f is a quasi-isometry there exist constants $k \geq 1$ and $c \geq 0$ such that for all $a, b \in \mathbb{Z}$,

$$\frac{1}{k} d_2(a, b) - c \leq d_3(f(a), f(b)) \leq k d_2(a, b) + c.$$

Let $D = \operatorname{lcm}(d_0, d_1, \ldots, d_m)$ and define $g(x) = Df(x)$. Note that g is a nonconstant polynomial with integer coefficients. For $a, b \in \mathbb{Z}$, Lemma 11 implies that

$$d_3(g(a), g(b)) = d_3(Df(a), Df(b)) \leq D d_3(f(a), f(b)).$$

It follows that

$$d_3(g(a), g(b)) \leq D k d_2(a, b) + Dc.$$

Thus $g(x)$ satisfies the upper bound of the quasi-isometric embedding condition with constants Dk and Dc. Therefore, by the contrapositive of Lemma 10, the set $A_g = \{d_3(g(a), 0) \mid a \in \mathbb{N}\}$ is bounded above. But by Lemma 12, the set A_g is unbounded above. Hence, f cannot be a quasi-isometry from C_2 to C_3. □

The results in this paper provide insight into Schwartz's question, but the question still remains open. In light of the discussion in the previous section, if the two spaces are not quasi-isometric, a natural way to proceed is to find a new quasi-isometric invariant that would distinguish between the two spaces. On the other hand, if those two spaces are quasi-isometric, finding a map that is a quasi-isometry would be one way to proceed. We are looking forward to seeing further investigations into answering this question.

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