Student Projects

Isolation and Contentment in Segregation Games with Three Types

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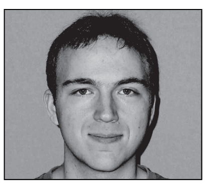

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Abstract

In *Individual Strategy and Social Structure*, Young describes a particular version of The Segregation Game. This paper begins to extend the work of Young to configurations involving three types of individuals. Seven scenarios representing different possible biases amongst the three types are identified, and contentment levels at equilibrium for each scenario are investigated. Surprisingly, holding biases against other types increases the likelihood of being isolated.

Introduction

Recently, an article in *The Economist* discussed a tendency for Americans to move from places with less like-minded people to places with more like-minded people [1]. One example showed that in 1976, when Jimmy Carter won the Presidency with 50.1 percent of the popular vote, 26.8 percent of Americans were in landslide counties, whereas during the 2004 Presidential Election, 48.3 percent lived in landslide counties.

To model these tendencies, Peyton Young uses a "segregation game" in his 1998 book, *Individual Strategy and Social Structure*, where two different types of people randomly switch places with each other in order to gain more neighbors that are like themselves [2]. If one individual improved its position and the other improved or retained its position, then the switch was made.

Young eventually proved that in a configuration with two different types of individuals, after all possible trades were made, no individual would be isolated: each would have at least one neighbor of the same type. Our research focuses on whether this holds true when configurations contain three different types of individuals. In seven possible scenarios, we answer this question and also determine the contentment level for a configuration at equilibrium under each of the scenarios.

Background

Definitions

A **configuration** is a circular sequence of individuals that contains at least two individuals of each type. We use circular configurations just as Young did so that we don't have to worry about the end conditions inherent in linear configurations. A **string** is a group of consecutive individuals within a configuration. An **alternating string** is a string where members of two types alternate. The **contentment level** of an individual is a value determined by the individual's neighbors and the individual's biases. This is discussed further in section 2.2. Also, the **contentment level** of a configuration is the minimum contentment level of any individual within a configuration. A **trade** occurs when two individuals exchange positions in the configuration so that one individual increases contentment level and the other does not decrease contentment level. A configuration is at **equilibrium** if no trades exist.

The Seven Scenarios

In configurations containing three different individual types, each individual will prefer neighbors of their own type, as they did in Young's work. The individual may also prefer neighbors of one of the other types over the second, resulting in a ranking of neighborhood patterns. Unlike Schelling, who looked at multiple neighbors to the left and right, we will only look at the neighbors directly to the left and right of an individual [3]. These preferences lead to seven different scenarios in which the contentment level of an individual is determined by its neighbors. In particular, $f_i(XYZ)$ is Y's contentment level under scenario i with a left neighbor X and a right neighbor Z.

In **Scenario 1**, neither A, B, nor C hold a bias against anyone else.

$$\begin{aligned}
f_1(AAA) &= f_1(BBB) = f_1(CCC) = 5 \\
f_1(AAB) &= f_1(AAC) = f_1(BBA) = f_1(BBC) \\
&= f_1(CCA) = f_1(CCB) = 4 \\
f_1(BAC) &= f_1(BAB) = f_1(CAC) = f_1(ABC) = f_1(ABA) \\
&= f_1(CBC) = f_1(ACB) = f_1(ACA) = f_1(BCB) = 3
\end{aligned}$$

In **Scenario 2**, A's prefer B's as neighbors, but neither B's nor C's hold any bias.

$$\begin{aligned}
f_2(AAA) &= f_2(BBB) = f_2(CCC) = 5 \\
f_2(AAB) &= f_2(BBA) = f_2(BBC) = f_2(CCA) = f_2(CCB) = 4 \\
f_2(AAC) &= f_2(BAB) = f_2(ABC) = f_2(ACB) = f_2(ABA) \\
&= f_2(ACA) = f_2(CBC) = f_2(BCB) = 3 \\
f_2(BAC) &= 2 \\
f_2(CAC) &= 1
\end{aligned}$$

In **Scenario 3**, A's prefer B's as neighbors, B's prefer C's, but C's have no bias.

$$\begin{aligned}
f_3(AAA) &= f_3(BBB) = f_3(CCC) = 5 \\
f_3(AAB) &= f_3(BBC) = f_3(CCA) = f_3(CCB) = 4 \\
f_3(AAC) &= f_3(BAB) = f_3(BBA) = f_3(CBC) \\
&= f_3(ACB) = f_3(ACA) = f_3(BCB) = 3 \\
f_3(BAC) &= f_3(ABC) = 2 \\
f_3(CAC) &= f_3(ABA) = 1
\end{aligned}$$

In **Scenario 4**, A's prefer B's as neighbors, B's prefer A's as neighbors, but C's have no bias.

$$\begin{aligned}
f_4(AAA) &= f_4(BBB) = f_4(CCC) = 5 \\
f_4(AAB) &= f_4(BBA) = f_4(CCA) = f_4(CCB) = 4 \\
f_4(AAC) &= f_4(BAB) = f_4(BBC) = f_4(ABA) \\
&= f_4(ACB) = f_4(ACA) = f_4(BCB) = 3 \\
f_4(BAC) &= f_4(ABC) = 2 \\
f_4(CAC) &= f_4(CBC) = 1
\end{aligned}$$

In **Scenario 5**, A's prefer C's as neighbors, B's also prefer C's as neighbors, but C's have no bias.

$$\begin{aligned}
f_5(AAA) &= f_5(BBB) = f_5(CCC) = 5 \\
f_5(AAC) &= f_5(BBC) = f_5(CCA) = f_5(CCB) = 4 \\
f_5(AAB) &= f_5(BBA) = f_5(CAC) = f_5(CBC) \\
&= f_5(ACB) = f_5(ACA) = f_5(BCB) = 3 \\
f_5(BAC) &= f_5(CBA) = 2 \\
f_5(BAB) &= f_5(ABA) = 1
\end{aligned}$$

In **Scenario 6**, A's prefer B's as neighbors, B's prefer C's as neighbors, and C's prefer A's.

$$\begin{aligned}
f_6(AAA) &= f_6(BBB) = f_6(CCC) = 5 \\
f_6(AAB) &= f_6(BBC) = f_6(CCA) = 4 \\
f_6(AAC) &= f_6(BBA) = f_6(CCB) \\
&= f_6(BAB) = f_6(CBC) = f_6(ACA) = 3 \\
f_6(BAC) &= f_6(CBA) = f_6(ACB) = 2 \\
f_6(CAC) &= f_6(ABA) = f_6(BCB) = 1
\end{aligned}$$

In **Scenario 7**, A's prefer B's as neighbors, B's prefer A's, and C's also prefer A's.

$$\begin{aligned}
f_7(AAA) &= f_7(BBB) = f_7(CCC) = 5 \\
f_7(AAB) &= f_7(BBA) = f_7(CCA) = 4 \\
f_7(AAC) &= f_7(BBC) = f_7(CCB) \\
&= f_7(BAB) = f_7(ABA) = f_7(ACA) = 3 \\
f_7(BAC) &= f_7(ABC) = f_7(ACB) = 2 \\
f_7(CAC) &= f_7(CBC) = f_7(BCB) = 1
\end{aligned}$$

Of course, other scenarios are possible. For example, in scenario 3, we decided that $f_3(AAC) = f_3(BAB) = 3$. If instead we decided that $f_3(AAC) = 4$ and $f_3(BAB) = 3$, then the maximum contentment level would be 6 instead of 5.

Results by Scenario

This section shows the minimum and maximum contentment levels of configurations at equilibrium in each of the seven scenarios. It will also describe some surprising results about isolation.

In the following proofs, we will use two specific symbols: the dash (–) and the ellipse (...). The long dash indicates a string of individuals that can be anything. For example, if we are interested in starting with a particular individual and identifying the individuals immediately following, we will write the string like this: $ABC-$, $BAB-$, etc. The ellipse indicates that there are no other individuals of a different type between two individuals of the same type. For example, no B's or C's would be present in the string $A \ldots A$.

First, though, we prove a lemma concerning alternating strings, important because it demonstrates that alternating strings will sort themselves out automatically in the other proofs, so we need not worry about them.

Lemma 4. *At an equilibrium, no alternating strings exist.*

Proof. By way of contrapositive, consider the configuration xy_1x_1yx-. A favorable trade between y_1 and x_1 exists. Therefore the configuration is not at equilibrium. □

Scenario 1

In this scenario, no individual is isolated in a configuration at equilibrium since the minimum contentment level for these configurations is 4. This will change, however, as the scenarios become more complex.

Theorem 5. *If a Scenario 1 configuration is at equilibrium, then it has a contentment level of at least 4.*

Proof. Suppose that there exists an individual whose contentment level at equilibrium is 3.

Case 1: Suppose A has two C's as neighbors. In the configuration

$$A \ldots AB_1B \ldots BC_1C \ldots CA_1C \ldots C-,$$

A_1 and C_1 can make a trade since A_1's contentment level stays at 3, and C_1's contentment level moves from 4 to 5. In the other possible configuration,

$$B \ldots BA \ldots AC_1C \ldots CA_1C \ldots C-,$$

A_1 and C_1 can also make a trade since A_1's contentment level moves from 3 to 4, and C_1's contentment level moves from 4 to 5. By symmetry, the minimum contentment level of an A with two B's as neighbors or a type B or C individual is 4.

Case 2: Suppose A has one B and one C as neighbors. In the configuration

$$A \ldots AC_1C \ldots CB \ldots BA_1C \ldots C-,$$

A_1 and C_1 can make a trade since A_1's contentment level moves from 3 to 4, and C_1's contentment level stays at 4. In the other possible configuration,

$$C \ldots CA \ldots AB_1B \ldots BA_1C \ldots C-,$$

A_1 and B_1 can make a trade since A_1's contentment level moves from 3 to 4, and B_1's contentment level stays at 4.

By symmetry, the minimum contentment level of a type B or C individual is 4. □

Scenario 2

In Scenario 2, the configuration $BBAABBCC$ has a contentment level of 4, which is the best any configuration could have; a configuration cannot have a contentment level of 5 even though individuals can. However, the following theorem shows that the minimum contentment level for a Scenario 2 configuration at equilibrium is 3.

Theorem 6. *If a Scenario 2 configuration is at equilibrium, then it has a contentment level of at least 3.*

Proof. Suppose without loss of generality that there exists an A such that A's contentment level is 1 or 2. This leads to two possible cases:

Case 1: Suppose A has two C's as neighbors and hence a contentment level of 1. In the configuration

$$A \dots AB \dots BC_1C \dots CA_1C \dots C-,$$

A_1 and C_1 can make a trade since A_1's contentment level moves from a 1 to 2, and C_1's contentment level moves from a 4 to 5. In the other possible configuration,

$$B \dots BA \dots AC_1C \dots CA_1C \dots C-,$$

A_1 and C_1 can make a trade since A_1's contentment level moves from a 1 to 3, and C_1's contentment level moves from a 4 to 5.

Case 2: Suppose that A has a B and a C neighbor and hence a contentment level of 2. In the configuration

$$A \dots AC_1C \dots CB \dots BA_1C \dots C-,$$

A_1 and C_1 can make a trade since A_1's contentment level moves from a 2 to 3, and C_1's contentment stays at 4. In the other possible configuration,

$$C \dots CC_1A \dots AB \dots BA_1C \dots C-,$$

A_1 and C_1 can make a trade since A_1's contentment level moves from a 2 to 3, and C_1's contentment stays at 4. □

Interestingly, an individual of type A can be left in isolation when a Scenario 2 configuration is at equilibrium, as the configuration CCBBABBCCAAA shows. This phenomenon will be seen again in the other scenarios.

Scenario 3

In Scenario 3, the minimum contentment level for a configuration decreases to 2 overall by the configuration $BBAACCBAA$. While this is different from Scenario 2, it is similar to Scenario 2 in that an individual of type A can be left in isolation in a configuration at equilibrium. For example, the configuration $CCBBA_1BBCCAA$ is at equilibrium, and A_1 is isolated. However, the configuration $CCAAB_1AACCBB$, shows that B can also be isolated in these scenarios.

Theorem 7. *If a Scenario 3 configuration is at equilibrium, then it has a contentment level of at least 2.*

Proof. Suppose A has contentment level 1 i.e. string $CAC-$. In the configuration

$$A \dots AB_1B \dots BC \dots CA_1C \dots C-,$$

A_1 and B_1 can make a trade, since A_1's contentment level moves from 1 to 4, and B_1's contentment level stays at 3. In the other possible configuration,

$$B \dots BA \dots AC_1C \dots CA_1C \dots C-,$$

A_1 and C_1 can make a trade, since A_1's contentment level moves from 1 to 3, and C_1's contentment level moves from a 4 to 5. Now suppose that B has a contentment level of 1. The configuration $ABA-$ follows similarly. In the configuration

$$B \dots BC \dots CA_1A \dots AB_1A \dots A-,$$

A_1 and B_1 can make a trade, and in the configuration

$$C \dots CB \dots BA_1A \dots AB_1A \dots A-,$$

A_1 and B_1 can again make a trade.

To construct a configuration with a contentment level of 4, each type A individual must be surrounded by other A's or B's, and each B must be surrounded by B's or C's, and these conditions are mutually exclusive. □

Scenario 4

In Scenario 4, the minimum contentment level for a configuration returns to 3. Again, an individual of type A can be left in isolation in a configuration at equilibrium. For example, the configuration $CCBBA_1BBCCAA$ is at equilibrium, and A_1 is isolated. However, the configuration $CCAAB_1AACCBB$, shows that B can also be isolated in these scenarios.

Theorem 8. *If a Scenario 4 configuration is at equilibrium, then it has a contentment level of exactly 3.*

Proof. Suppose without loss of generality that there is an individual with contentment level 1 or 2, again leading to two possible cases. After finding trades in these cases, we will show that it is impossible to construct a configuration with a contentment level of 4.

Case 1: Suppose an A has contentment level 1. In the configuration

$$A \dots AB \dots BC_1C \dots CA_1C \dots C-,$$

A_1 and C_1 can make a trade, since A_1's contentment level moves from a 1 to 2, and C_1's contentment level moves from a 4 to 5. In the other possible configuration,

$$B \dots BA \dots AC_1C \dots CA_1C \dots C-,$$

A_1 and C_1 can make a trade, since A_1's contentment level moves from 1 to 3, and C_1's contentment level moves from a 4 to 5. The configuration $CBC-$ follows by symmetry.

Case 2: Suppose an A has contentment level 2. In the configuration

$$A \ldots AC_1C \ldots CB \ldots BA_1C \ldots C-,$$

A_1 and C_1 can make a trade, since A_1's contentment level moves from a 2 to 3, and C_1's contentment level stays at 4. In the other possible configuration,

$$C \ldots CC_1A \ldots AB \ldots BA_1C \ldots C-,$$

A_1 and C_1 can make a favorable trade, since A_1's contentment level moves from 2 to 3, and C_1's contentment level stays at a 4. $ABC-$ follows by symmetry.

To construct a configuration with a contentment level of 4, each type A individual must be surrounded by A's or B's, and each type B individual must be surrounded by B's or A's, but this is impossible because at least one individual of type A or B would have a C next to it, decreasing its contentment level to 3. □

Scenario 5

While a Scenario 5 configuration exists, $BBCCA_1CCBBAA$, where an individual is in isolation in a configuration at equilibrium, unlike Scenarios 3 and 4, it is possible to construct a configuration with a contentment level of 4 at equilibrium: $CAACCBBC$. The theorem will show that configurations at equilibrium have a minimum contentment level of 3.

Theorem 9. *If a Scenario 5 configuration is at equilibrium, then it has a contentment level of at least 3.*

Proof. Suppose without loss of generality that there exists an individual with contentment level 1 or 2. This leads to two possible cases.

Case 1: Suppose an individual of type A has contentment level 1. In the configuration

$$A \ldots AC \ldots CB_1B \ldots BA_1B \ldots B-,$$

A_1 and B_1 can make a trade, since A_1's contentment level moves from a 1 to 2, and B_1's contentment level moves from a 4 to 5. In the other possible configuration,

$$C \ldots CA \ldots AB_1B \ldots BA_1B \ldots B-,$$

A_1 and B_1 can make a trade, since A_1's contentment level moves from 1 to 3, and B_1's contentment level moves from a 3 to 5. The configuration $ABA-$ follows by symmetry.

Case 2: Suppose an individual of type A has a contentment level of 2. In the configuration

$$A \ldots AC_1C \ldots CB \ldots BA_1C \ldots C-,$$

A_1 and C_1 can make a trade, since A_1's contentment level moves from a 2 to 4, and C_1's contentment level stays at 4. In the other possible configuration,

$$C \ldots CC_1A \ldots AB \ldots BA_1C \ldots C-,$$

A_1 and C_1 can make a trade, since A_1's contentment level moves from 2 to 4, and C_1's contentment level stays at a 4. The configuration $ABC-$ follows by symmetry. □

Scenarios 6 and 7

Additional bias is added in Scenarios 6 and 7. In these scenarios, it is impossible to construct a configuration with a contentment level of 4, and but is possible to construct a configuration with a contentment level of 3, $AAAABBBBCCCC$, the minimum contentment level for Scenarios 6 and 7 configurations is 2, since $AACCBBACC$ is at equilibrium under Scenario 6, and $CCAABBACC$ is at equilibrium under Scenario 7, both with contentment levels of 2.

Theorem 10. *If a Scenario 6 configuration is at equilibrium, then it has a contentment level of at least 2.*

Proof. Suppose that there exists an individual of type A with contentment level 1. In the configuration

$$A \ldots AB \ldots BC_1C \ldots CA_1C \ldots C-,$$

A_1 and C_1 can make a trade, since A_1's contentment level moves from a 1 to 2, and C_1's contentment level moves from a 4 to 5. In the other possible configuration,

$$B \ldots BA \ldots AC_1C \ldots CA_1C \ldots C-,$$

A_1 and C_1 can make a trade, since A_1's contentment level moves from 1 to 3, and B_1's contentment level moves from a 4 to 5. The configurations $ABA-$ and $BCB-$ follow by symmetry. □

Theorem 11. *If a Scenario 7 configuration is at equilibrium, then it has a contentment level of at least 2.*

Proof. Suppose that there is an individual of type A with contentment level 1. In the configuration

$$A \ldots AB \ldots BC_1C \ldots CA_1C \ldots C-,$$

A_1 and C_1 can make a trade, since A_1's contentment level moves from a 1 to 2, and C_1's contentment level moves from a 3 to 5. In the other possible configuration,

$$B \ldots BA \ldots AC_1C \ldots CA_1C \ldots C-,$$

A_1 and C_1 can make a trade, since A_1's contentment level moves from 1 to 3, and B_1's contentment level moves from a 4 to 5. The configurations $CBC-$ and $BCB-$ follow by symmetry. □

Finally, all three types of individuals can be isolated in Scenarios 6 and 7, since $CCBBABBCCAA$, $AACCBCCAABB$, and $BBAACAABBCC$ are in equilibrium and have isolated individuals.

Concluding Remarks

As expected, as the number of biases increases, the lower the minimum contentment level for a configuration. The most interesting discovery made, though, is that when an individual shows bias, he or she opens up the possibility of being isolated, whereas the possibility of being isolated is much lower when not showing bias. We had thought that individuals who do not show bias are guaranteed to have at least one neighbor like themselves at equilibrium. However, this is not true in all scenarios.

Suppose that we are in one of Scenarios 2-5, C has no bias, and C has two A's as neighbors. In the configuration

$$B \dots BC \dots CA_1A \dots AC_1A \dots A-,$$

A_1 and C_1 can make a trade since both A_1 and C_1's contentment levels would improve in any of the scenarios. In the other possible configuration,

$$C \dots CB \dots BA_1A \dots AC_1A \dots A-,$$

A_1 and C_1 can again make a trade since A_1's contentment level moves to 5, and C_1's contentment level cannot move lower than 3 in any of the scenarios. Similarly, this follows if C has two B's as neighbors.

However, suppose under Scenario 5 that C has one A and one B as neighbors. The C_1 in the configuration

$$BBCCAAC_1BB$$

is at equilibrium and is isolated. Further, in the configuration

$$CCBBAAC_1BB,$$

C_1 is isolated in both Scenario 3 and 5, and C shows no bias in either of the configurations. Similar to Schelling's work, we intend to expand our research to two-dimensional models, but with three individual types [3]. This could perhaps represent a more accurate model, but a new set of scenarios and rules would need to be developed.

References

[1] The Big Sort, The Economist, 2008, June 21.

[2] H. Young, Individual Strategy and Social Structure, Princeton University, Press (2001).

[3] T. Schelling, Dynamic Models of Segregation, *Journal of Mathematical Sociology* **1** (1971) 143-186.