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Markov Chains of Chord Progressions

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Abstract We explore an area in which mathematics and music overlap. We analyze the chord progressions of compositions by four well-known composers, namely Palestrina, Bach, Mozart, and Beethoven, from different musical periods in history using Markov Chain Analysis. We analyze the resulting steady state vectors and find they agree with the chord progression models of each musical era.

Introduction & Definitions

One of the first people to make a connection between mathematics and music was Pythagoras of Samos, nearly 2500 years ago. Since then, many mathematicians have explored areas of overlap between mathematics and music. In this project, we explored an overlap between linear algebra and music. Our goal is to find chord progressions that are common to a specific collection of music, composer, or era.

To understand chord progressions, however, we must first understand the musical scale and its respective chords. A musical scale consists of seven notes and repeats at the octave, e.g., the C Major Scale is C-D-E-F-G-A-B-(C)-.... The notes of a musical scale are numbered by their steps from the root of the scale. In a C Major Scale, the root is C, the second D, the third E, and so on to the seventh B, at which point the scale starts over again one octave higher.

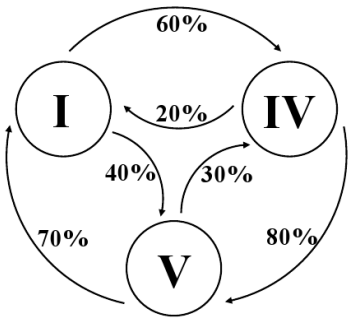
A chord is three or more notes played simultaneously. In our work, we consider seven chords defined on a musical scale. The C Major scale, for example, consists of the I chord (C-E-G), ii (D-F-A), iii (E-G-B), IV (F-A-C), V (G-B-D), vi (A-C-E), and vii° (B-D-F). An upper case Roman numeral indicates a major chord whereas a

lower case Roman numeral indicates a minor chord, and a superscript ° indicates a diminished chord (flat fifth).

A chord progression, then, is the movement from one chord to another in a musical scale. For example, I-IV-V-I and I-V-vi-IV are two very common chord progressions used in pop music [1].

A Markov chain is a mathematical system that goes through transitions from one state to another, in which the next state is dependent only on the current state and not on the sequence of events that preceded it [2].

For example, suppose a piece of music has the following probabilities changing from one chord to another,



We would represent this situation mathematically with the following Initial State matrix:

$$\begin{array}{c} \\ \text{I} \\ \text{IV} \\ \text{V} \end{array} \begin{array}{c} \begin{array}{ccc} \text{I} & \text{IV} & \text{V} \end{array} \\ \begin{pmatrix} 0 & .60 & .40 \\ .20 & 0 & .80 \\ .70 & .30 & 0 \end{pmatrix} \end{array}$$

Using Markov Chain Analysis, we find that the steady state is

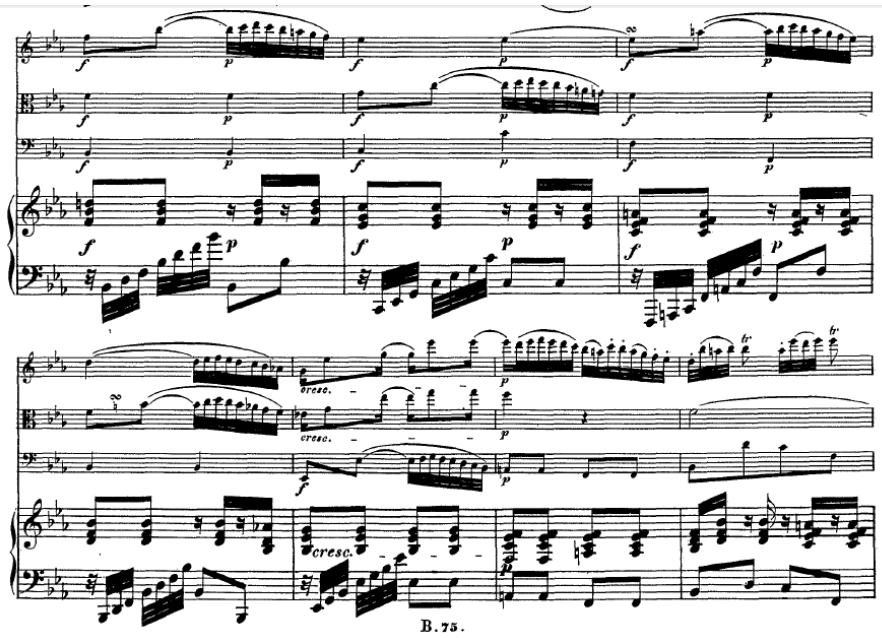
$$\begin{array}{ccc} \text{I} & \text{IV} & \text{V} \end{array}$$
$$\begin{pmatrix} .322 & .305 & .373 \end{pmatrix}$$

which tells us that in the long run we expect 32.2% of chord changes to progress to I chords, 30.5% to IV chords, and 37.3% to V chords.

Our Data

We chose compositions from Palestrina, Bach, Mozart, and Beethoven because we wanted to analyze chord progressions from the different musical periods: Renaissance (1400 - 1600), Baroque (1600 - 1760), Classical (1750 - 1830), and Romantic (1815 - 1910), respectively.

Below is a seven-measure excerpt from Beethoven's *Piano Quartet No. 1.*



Musical analysis shows the chord progression to be B♭-Cm-F-B♭-E♭-F-(B♭-F). For our research, it is useful to interpret the chord progression in regards to the composition's key signature - the key of E-flat major - which gives us V-vi-ii-V-I-ii-(V-ii). Its initial state matrix is found by tallying up each progression from one chord to the next; in this case, V-vi, vi-ii, ii-V, V-I, and so on.

Using data sheets on chord progressions on Bach and Mozart from Tymoczko [3], who collected data to create chord transition matrices previously [4], and data we collected ourselves on Palestrina and Beethoven, we gathered the initial state matrices shown in Table 1. Our data was collected from sample sizes of at least 300 measures on each of Palestrina and Beethoven, while the data collected by Tymoczko used at least 1000 chord transitions each.

Each entry of the matrices in Table 1 represents the probability that the chord in the column immediately follows the chord in the row. For example, in the first row of the Bach Minor matrix, the .41 means that there is a 41% chance that any I chord is followed by a V chord. Note that each row sums to 1, or 100%.

Bach Minor	I	ii	iii	IV	V	vi	vii°
I	0	.18	.01	.20	.41	.09	.12
ii	.01	0	.03	0	.89	0	.07
iii	.06	.06	0	.25	.19	.31	.13
IV	.22	.14	0	0	.48	0	.15
V	.80	0	.02	.06	0	.10	.02
vi	.03	.54	.03	.14	.19	0	.08
vii°	.81	0	.01	.03	.15	0	0

Bach Major	I	ii	iii	IV	V	vi	vii°
I	0	.15	.01	.28	.41	.09	.06
ii	.01	0	0	0	.71	.01	.25
iii	.03	.03	0	.52	.06	.32	.03
IV	.22	.13	0	0	.39	.02	.23
V	.82	.01	0	.07	0	.09	0
vi	.15	.29	.05	.11	.32	0	.09
vii°	.91	0	.01	.02	.04	.03	0

Mozart Minor	I	ii	iii	IV	V	vi	vii°
I	0	.08	0	.07	.68	.06	.11
ii	.37	0	0	0	.46	0	.17
iii	0	0	0	1	0	0	0
IV	.42	.10	0	0	.39	0	.09
V	.82	0	0	.05	0	.07	.05
vi	.14	.51	0	.16	.05	0	.14
vii°	.76	.01	0	0	.23	0	0

Mozart Major	I	ii	iii	IV	V	vi	vii°
I	0	.13	0	.15	.62	.05	.05
ii	.49	0	.01	0	.40	.01	.09
iii	.67	0	0	0	0	.33	0
IV	.64	.14	0	0	.15	0	.07
V	.94	0	0	.01	0	.04	.01
vi	.11	.51	0	.14	.20	0	.04
vii°	.82	0	.01	.01	.16	0	0

Palestrina	I	ii	iii	IV	V	vi	vii°
I	0	.15	.13	.28	.14	.22	.08
ii	.08	0	.15	.13	.28	.14	.22
iii	.22	.08	0	.15	.13	.28	.14
IV	.14	.22	.08	0	.15	.13	.28
V	.28	.14	.22	.08	0	.15	.13
vi	.13	.28	.14	.22	.08	0	.15
vii°	.15	.13	.28	.14	.22	.08	0

Beethoven	I	ii	iii	IV	V	vi	vii°
I	0	.10	.01	.13	.52	.02	.22
ii	.06	0	.02	0	.87	0	.05
iii	0	0	0	0	.67	.33	0
IV	.33	.03	.07	0	.40	.03	.13
V	.56	.22	.01	.04	0	.07	.11
vi	.06	.44	0	.06	.11	0	.33
vii°	.80	0	0	.03	.17	0	0

Table 1: Transition matrices for Palestrina, Bach, Mozart, and Beethoven.

Data Analysis

We find the steady state vectors to be...

	I	ii	iii	IV	V	vi	vii°
Bach Minor	.333	.109	.015	.010	.306	.063	.075
Bach Major	.344	.089	.016	.131	.273	.067	.080
Mozart Minor	.400	.066	0	.056	.345	.049	.084
Mozart Major	.435	.086	.001	.073	.330	.037	.038
Palestrina	.143	.143	.143	.143	.143	.143	.143
Beethoven	.317	.120	.012	.060	.328	.034	.130

For example, the results for Bach Minor show that there is a 33.3% probability of progressing to the I chord from the previous chord, 10.9% to ii, 1.5% to iii, 1.0% to IV, 30.6% to V, 6.3% to vi, and 7.5% to vii°.

Our analysis shows that that progressions to the I and V chords are most probable in all compositions (except Palestrina), whereas progressions to the iii chord are the least probable.

Noteworthy Results

Composer	Most Probable Chord(s)	Least Probable Chord(s)
Palestrina	Equal Probability (14.3%)	Equal Probability (14.3%)
Bach	**I** (33.9%) and **V** (29.0%)	**iii** (1.5%)
Mozart	**I** (41.8%) and **V** (33.8%)	**iii** (0%)
Beethoven	**V** (32.8%) and **I** (31.7%)	**iii** (1.2%)

Music from the Renaissance Period emphasized the self, and introduced the concept of chord progressions. The Church, which emphasized the divine, disapproved of the Renaissance Period because of its contrasting view. Palestrina, a composer of sacred music, composed in a manner the Church approved of - and this was when chord progressions did not exist, prior to the Renaissance Period. Our results on Palestrina's work correlate with this musical history because there is an equal probability of progressing to any chord; i.e., it suggests that Palestrina did not follow any chord progression model.

Our results for Bach and Mozart agree with the Baroque and Classical eras, respectively, because they show the dominance of the I and V chord - where the I chord is most probable - and the lack of the iii chord. Beethoven's *Piano Quartet*

No. 1, which was composed in 1783, during the Classical Period, also conforms to this trend. The late classical period was marked by compositions that were internally more complex. We see this demonstrated by the V chord becoming most probable in Beethoven's work due to a 6 to 9% drop in the relative dominance of the I chord over the V chord.

Romantic music broke down formal structures, such as chord progression models, from the Classical Period. Beethoven straddles the Classical and Romantic periods so we only begin to see changes in the steady state matrix compared to Mozart. If we were to analyze later compositions from the Romantic era, we would expect a steady state vector that shows no trend. In particular, we would expect the dominance of the I and V chord and the lack of the iii chord from the Classical Period to disappear almost entirely.

Acknowledgments & Directions for Further Study

Our research can serve as a launching pad for chord and Markov Chain Analysis for any number of compositions and composers. With enough data, it would perhaps be possible to show the changes happening gradually in musical eras, say from the Classical Period to the Romantic Era, and therefore defining features of each musical time period.

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