Mathematical Modeling of the Nautilus Shell

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Ashley Allen is a Ball State senior majoring in Actuarial Science. She was a member of the Math Department's UniverCity 2002 project in her junior year.

UniverCity is a now biennial event that Ball State University and the City of Muncie sponsor to benefit the community. This year's theme was *Celebrate Community* and the symbol was the nautilus shell. UniverCity provided this statement: "The magnificent Nautilus Shell was chosen as a symbol for the UniverCity 2002 festival at Ball State University. With the theme *Celebrate Community*, UniverCity 2002 was focused upon cooperation, collaboration and connections among and between people. The Nautilus Shell found a strong fit with our theme. This shell, like a community, gains its strength through the harmonious coming together of its many individual parts. It grows outward from a strong center, like our towns and cities. Its spiral geometry evokes order and proportion. Its rainbow of color reflects the diversity and richness so essential to a healthy, vibrant community. Lastly, the simple outer appearance of the Nautilus belies its phenomenal inner beauty—symbolic of the inner beauty and magic so inherent to people and communities." Departments were invited to do a project to share with the community. The Department of Mathematical Sciences chose to do a mathematical modeling of the shape of the nautilus shell. Our team consisted of Josh Drew and me, and our faculty advisor was Dr. Giray Ökten. We used the following polar equation to model the spiral shape of the shell:

$$r = a * e^{\theta \cot(b)}$$

The curves of the spiral are called equiangular (or logarithmic) spirals. The smaller the constant a, the tighter the spiral becomes. The constant b is the angle between the radial line and the tangent line. This is consistent with every turn. The picture below is the spiral of a real nautilus shell. It shows the radial line and the tangent lines. The arrow points to angle b. As the variable θ increases the spiral grows.
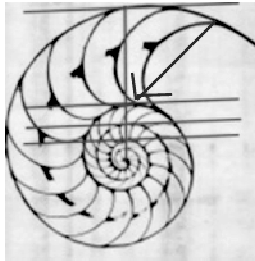
For a better understanding, interactive 2D and 3D simulations were created using mathematical software. Here is an example of the 3D picture, but this does not facilitate understanding as well as the 2D simulations.

3-dimensional computer simulation
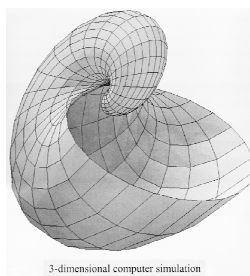
Below are the 2D simulations, with a and b changing. Specifically we chose $a = 1.2, b = 1.4$, $a = .8, b = 1.4$, and $a = .8, b = 1.45$, respectively:
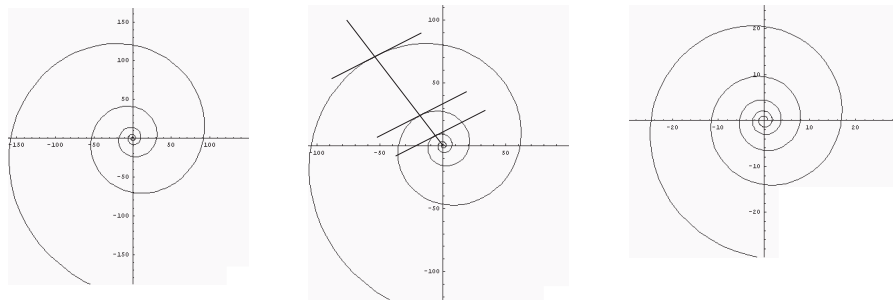
Interestingly enough, a fairly good approximation of the nautilus structure can be obtained by drawing squares whose sides are given by the Fibonacci numbers f_n (which are defined by the recursion $f_n = f_{n-1} + f_{n-2}$ with initial values $f_1 = 1$ and $f_2 = 1$) and aligning them as seen in the figure below. If we measure the growth of the spiral by measuring the growth of the sides of the squares, we will see that the growth ratio equals $\frac{f_n}{f_{n-1}}$, which approaches the golden ratio $\frac{1+\sqrt{5}}{2}$ as n grows without bound. The golden ratio can be observed throughout nature in star cluster nebulas, clouds, whirlpools, and even in the inner ear.
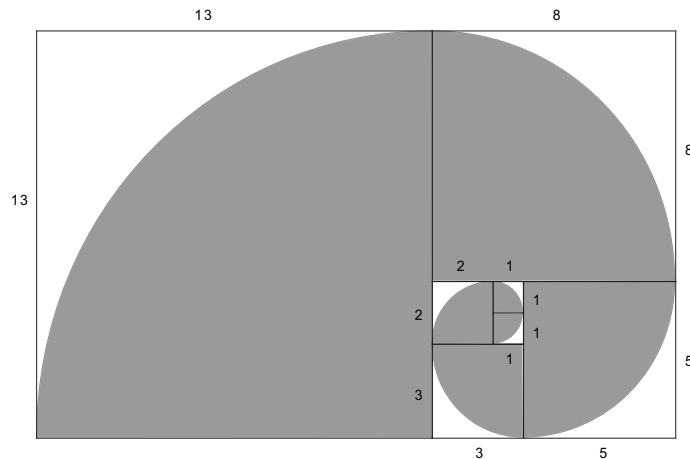
More information on the nautilus: The nautilus is a cephalopod, which is a class of the mollusk. After hatching, the nautilus is already 2-3 cm in diameter and has 7 chambers. As an adult, the shell will grow to be approximately 25-30 cm across. The chambered shell protects the soft animal and gives it buoyancy. The shell can withstand pressures in depths of 650 m below the ocean's surface. The chambers of the shell are separate, but each is connected to the others by a tube running through them. For the nautilus to move itself upward from the ocean floor, the tube pours gas into the chambers to float and liquid into the chambers to sink. As the nautilus grows, it forms a new chamber every lunar month. The actual animal occupies the outermost chamber. The nautilus lives in the Pacific Islands amongst Japan, Philippines, Micronesia, and as far as Fiji. As for the life and habits of the nautilus, they remain a mystery. However, there have been great strides in recent years to learn more about these animals.

For more fun with the nautilus spiral and Fibonacci sequence, visit the following web pages:

1. `http://is.dal.ca/~ceph/TCP/Npompil.html`

2. `http://www.vfs.com/~lwhitcom/3d.htm`

3. `http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/`
   `fibnat.html`