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Measuring Growth of Groups

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Abstract We present explicit calculations of growth functions and growth series of a variety of finitely generated groups, followed by some general results about those measures of growth. We provide a self-contained proof that the growth series of a group is rational if and only if the growth function satisfies a recurrence relation of finite depth. Our goal is to showcase the subject matter of the paper as a very accessible intersection of group theory, basic metric geometry, and combinatorics.

1 Introduction

The growth of finitely generated groups has been a much studied topic in recent years, especially since interest in this topic was rejuvenated in the 1980s with M. Gromov's characterization of groups of polynomial growth as virtually nilpotent. For a quick flavor of the subject matter and its connections, see [1].

For a finitely generated group, the (cumulative) growth function $\alpha(n)$ is defined as the number of group elements contained in a closed ball of radius n about the identity element, and the spherical growth function $\sigma(n)$ as the number of group elements contained in the boundary of the closed ball. We note that the group is finite if and only if $\alpha(n)$ is eventually constant, so we mainly focus on infinite (finitely generated) groups.

In the current paper we focus on explicit calculations of growth functions and growth series. The objective is to calculate these measures of growth directly whenever possible, and later compare some of these calculations with general results. As far as the group's asymptotic geometry is concerned, the exact forms of the growth functions are of less importance than the equivalence classes of its growth functions (see Definition 25); however, we feel that calculating the exact forms of the growth functions for some of these groups show interesting details of the geometry of the Cayley graphs of these groups. Moreover, this helps our aim of showcasing the subject matter of the paper as a very accessible intersection of group theory, basic metric geometry, and combinatorics.

The paper is organized as follows. Section 2 gives the preliminaries of viewing a finitely generated group as a metric space and defines the growth functions and series. Section 3 gives explicit calculations of growth functions and series for some commonly occurring groups. This section is one of the salient features of this paper, as explicit calculations for some of those cases are not easily found in the literature. Section 4 gives some general results about growth. While the general results in this section are well known, we have tried to give simple and self-contained proofs of the results starting from first principles. In this section we also give examples of distinct groups with the same growth function and ask a related question. Section 5 gives a self-contained proof of the fact that the growth series of a group is rational if and only if the growth function eventually satisfies a recurrence relation.

This work is based on an undergraduate research project, started when J. Preston was a student at Montana Tech. We have tried to keep the prerequisites of this exposition to a minimum, at the level of beginning material from undergraduate level first courses in group theory and metric spaces.

The authors wish to thank the anonymous referee for the very detailed comments. The referee's guidelines helped improve the exposition of this paper to a great extent.

2 Preliminaries

Any group can be represented in terms of a set S of symbols (called the generators) and a set R of relations between those symbols (called the relaters) [4]. We denote the group as $G = \langle S : R \rangle$. Technically, G is a quotient of the free group on the set S by the smallest normal subgroup of G containing R.

In geometric group theory, one usually explores asymptotic properties of infinite groups which have a finite generating set. Below we describe a way of defining a metric on a finitely generated group. See [5] for discussions on some of these concepts.

Definition 1. Let G be a group and S be a finite set that generates G. We assume $S = S^{-1} = \left\{ s^{-1} : s \in S \right\}$. We define the **word length** of g (norm of g) to be

$$\|g\| = \min \left\{ n : g = s_1 s_2 \cdots s_n, s_i \in S \right\}.$$

We define the **word metric** on G by setting

$$d_S(g_1, g_2) = \left\| g_1^{-1} g_2 \right\| = \min \left\{ n : g_1^{-1} g_2 = s_1 s_2 \cdots s_n, s_i \in S \right\}.$$

We note that the the metric described above takes only integral values. However, there is another way of associating a metric space with a finitely generated group, where the metric can take non-integral values. We will describe it below, and then we will see that these two metrics are essentially the same.

Definition 2. Given a group G and a finite generating set S, we define the **Cayley graph** $Cay(G, S)$ to be the metric graph such that the following conditions hold.

1. Vertices of the graph are the elements of G.
2. There is an edge connecting $g_1, g_2 \in G$ if and only if $g_1^{-1} g_2 \in S \cup S^{-1}$.
3. Each edge has length 1.

Taking each edge to be isometric to the interval $[0, 1]$, we can define a metric on $Cay(G, S)$ by declaring the distance between two points to be the infimum of the lengths of all paths between them.

For example, $Cay(\mathbb{Z}, \{1\})$ can be seen in Figure 1. Here any two vertices in $\mathbb{Z}$ (considered additively) are connected by an edge if they differ by 1. Similarly, in the case of $Cay(\mathbb{Z}, \{2, 3\})$ two vertices are joined by an edge if they differ by 2 or 3, as can be seen in Figure 2.

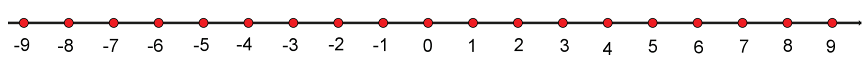

Figure 1: Cayley graph of $\mathbb{Z}$ with generating set $\{1\}$

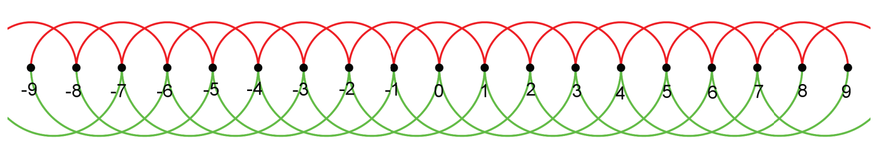

Figure 2: Cayley graph of $\mathbb{Z}$ with generating set $\{2, 3\}$

The above two ways of viewing a finitely generated group as a metric space are the same in the sense of the following notion of equivalence.

Definition 3. A function $f : (X, d_X) \to (Y, d_Y)$ is said to be a **quasi-isometric embedding** if there exist $\lambda, \mu > 0$ such that

$$\frac{1}{\lambda} d_X(x, x') - \mu \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + \mu.$$

Additionally, if f is coarsely surjective (i.e. there exists $D > 0$, such that for all $y \in Y$, there exists $x \in X$ such that $d_Y(f(x), y) < D$), then f is said to be a **quasi-isometry**.

We note that embedding a finitely generated group with the word metric into its Cayley graph is a quasi-isometry.

Now that we can look at a finitely generated group as a metric space, we can define its growth functions and growth series. For the next definition, recall that the closed ball of radius n (in the word metric) around the identity element $e \in G$ is the set $\bar{B}_n(e) = \{g \in G : d_S(g, e) \leq n\}$.

Definition 4. Let G be a finitely generated group and S be a finite generating set. We define the **growth function** $\alpha : \mathbb{N} \to \mathbb{N}$ as

$$\alpha(n) = \left|\bar{B}_n(e)\right|$$

for $n \in \mathbb{N}$. Note that as G is finitely generated, $\left|\bar{B}_n(e)\right|$ is always finite.

Definition 5. We define the **spherical growth function** as

$$\sigma(n) = \alpha(n) - \alpha(n-1).$$

for all $n \geq 1$. Also, we define $\sigma(0) = \alpha(0) = 1$.

The spherical growth function can also be thought of as the number of elements whose length is exactly n. Now we can define the corresponding growth series.

Definition 6. The **growth series** is defined as the formal power series

$$A(z) = \sum_{n=0}^{\infty} \alpha(n) z^n.$$

Definition 7. Similarly, we define the **spherical growth series** as the formal power series

$$S(z) = \sum_{n=0}^{\infty} \sigma(n) z^n.$$

Remark 8. *The power series that are introduced in Definitions 6 and 7 are "formal" power series - algebraic objects without any notion of convergence. More specifically, these are elements of the ring of formal power series with real coefficients* $\mathbb{R}[[x]]$*, where the addition and multiplication are defined similar to as in the ring of polynomials with real coefficients. In fact, the ring of* *polynomials with real coefficients $\mathbb{R}[x]$ is a subring of $\mathbb{R}[[x]]$. Formal power series are often used in combinatorics, in the technique of using "generating functions" as an efficient tool of dealing with infinite sequences of numbers. For a nice introduction to generating functions see [6].*

We use Cayley graphs to help us find growth functions and growth series. It is often easier to find the spherical growth function σ than the growth function α. It would be useful to somehow relate the two growth functions and the two growth series.

Theorem 9. *The following equalities hold.*

1. $\alpha(n) = 1 + \sum_{i=1}^{n} \sigma(i)$
2. $S(z) = (1-z)A(z)$

Proof. The proof of part 1 comes directly from the definitions of α and σ.

For proving part 2, from Definition 7 and Definition 5, we have

$$
\begin{aligned}
S(z) &= \sum_{n=0}^{\infty} \sigma(n) z^n \\
&= \alpha(0) + [\alpha(1) - \alpha(0)]\, z + [\alpha(2) - \alpha(1)]\, z^2 + \cdots + [\alpha(n) - \alpha(n-1)]\, z^n + \cdots \\
&= \big(\alpha(0) + \alpha(1)z + \alpha(2)z^2 + \alpha(3)z^3 + \cdots\big) - \big(\alpha(0)z + \alpha(1)z^2 + \alpha(2)z^3 + \cdots\big) \\
&= A(z) - zA(z) \\
&= (1-z)A(z).
\end{aligned}
$$

□

3 Examples

In this section, we calculate the growth functions and series of some commonly occurring groups. These are direct calculations, based on the definitions of the growth functions and series, and visualizations of the Cayley graphs of the corresponding groups in each case. We omit the sketches of most of the Cayley graphs, all of which are easily drawn.

Example 10. *Consider the group $\mathbb{Z}$ with standard generating set $\{1\}$. We can easily find the growth functions and growth series by looking at $Cay(\mathbb{Z}, \{1\})$ from Figure 1. From $Cay(\mathbb{Z}, \{1\})$ we can see that $\sigma(n) = 2$ for $n = 1, 2, \ldots$ as the number of elements of length n is always 2, thus from Theorem 9 we have $\alpha(n) = 2n + 1$ for $n = 1, 2, \ldots$. Now that we have α and σ, we can find A and S. We show this in detail to demonstrate the usual technique of finding these series. We have*

$$S(z) = \sum_{n=0}^{\infty} \sigma(n) z^n = 1 + \sum_{n=1}^{\infty} 2z^n = 1 + \frac{2z}{1-z} = \frac{1+z}{1-z}.$$

From Theorem 9, we have

$$\frac{1+z}{1-z} = (1-z)A(z) \text{ and so } A(z) = \frac{1+z}{(1-z)^2}.$$

Example 11. *Now, consider the same group $\mathbb{Z}$ but with generating set $\{2, 3\}$. The Cayley graph can be seen in Figure 2, where the curved lines above the horizontal represent the elements which differ by the 2 generator and the curved lines below the horizontal represent the elements which differ by the 3 generator. The values for $\alpha(1)$, $\alpha(2)$, $\alpha(3)$, and $\alpha(n)$ can be better seen in Figure 3.*

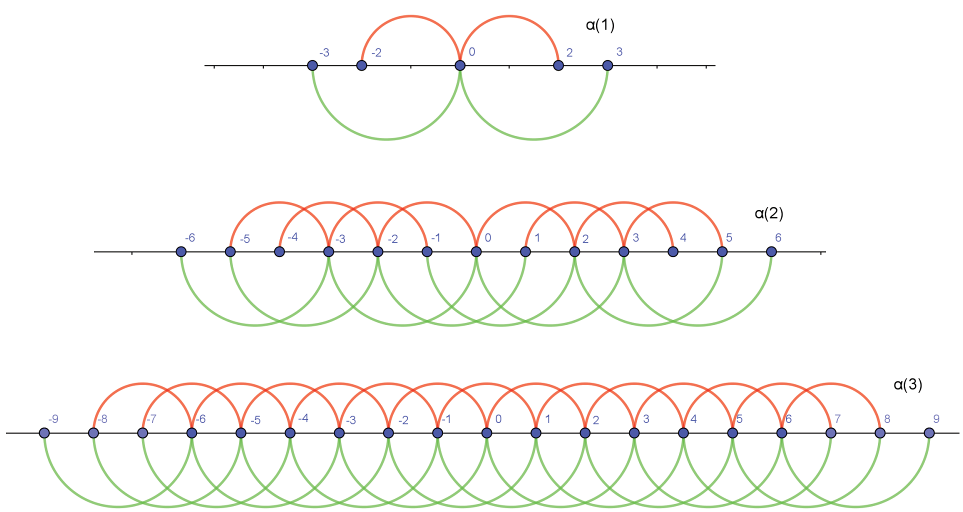

Figure 3: $\bar{B}_n(0)$ for $n = 1, 2, 3$. After $n = 2$ there is a clear pattern in the Cayley graph. We simply add the elements $\pm(3n+1), \pm(3n+2), \pm(3n+3)$.

For $n \geq 2$, every element from $-3n$ to $3n$ is in $\bar{B}_n(0)$. By adding the two generators to $3n$ we get the elements $3n+2$ and $3n+3$. We can also add the 2 generator to $3n-1$ to get $3n+1$. Similarly we can subtract the two generators from $-3n$ and subtract the 2 generator from $-3n+1$. Hence for $\bar{B}_{n+1}(0)$ we add the elements $\pm(3n+1), \pm(3n+2), \pm(3n+3)$. For our growth functions and series we have

$$\alpha(n) = \begin{cases} 1 & n = 0 \\ 5 & n = 1 \\ 6n+1 & n = 2,3,\dots \end{cases}$$

$$\sigma(n) = \begin{cases} 1 & n = 0 \\ 4 & n = 1 \\ 8 & n = 2 \\ 6 & n = 3,4,\dots \end{cases}$$

$$S(z) = \frac{1+3z+4z^2-2z^3}{1-z}$$

$$A(z) = \frac{1+3z+4z^2-2z^3}{(1-z)^2}.$$

Compare these results with $Cay(\mathbb{Z},\{1\})$ *in Example 10. Note that the growth function* α *for* $\mathbb{Z}$ *with standard generator* $\{1\}$ *is different than the one for* $\mathbb{Z}$ *with generators* $\{2,3\}$*, but both are (piecewise) linear.*

Remark 12. *One can create infinitely many examples by considering* $\mathbb{Z}$ *with generating set* $\{p,q\}$*, where* p,q *are primes. Assume* $p<q$*. Since* p,q *are prime,* $(p,q)=1$*. Then, there exists* $s,t\in\mathbb{Z}$*, such that* $sp+tq=1$*. We conjecture that in such a case we have the following formula for the spherical growth function (where* s *and* t *are such that* $|s|+|t|$ *is minimal):*

$$\sigma(n) = \begin{cases} 1, & n = 0 \\ 4n, & n = 1,2,\dots,|s|+|t| \\ 2(p+q-2), & n = |s|+|t|+1,\dots \end{cases}.$$

It will be interesting to try to get growth functions for $\mathbb{Z}$ *with generating set* $\{p,q\}$ *(where* p,q *are primes), without* $|s|,|t|$ *explicitly appearing in the expressions.*

Example 13. *Consider the group* $\mathbb{Z}^2=\mathbb{Z}\oplus\mathbb{Z}$ *with standard generating set* $S=\{(1,0),(0,1)\}$*. One can see that the elements in the sphere* $\bar{B}_n((0,0))\setminus\bar{B}_{n-1}((0,0))$ *are* $(0,\pm n),(\pm1,\pm(n-1)),(\pm2,\pm(n-2)),\dots,(\pm n,0)$ *so there are* $2+4+4+\cdots+4+2=4n$ *elements. Hence* $\sigma(n)=4n$ *for* $n=1,2,\dots$*. The spherical growth function can be better visualized in Figure 4.*

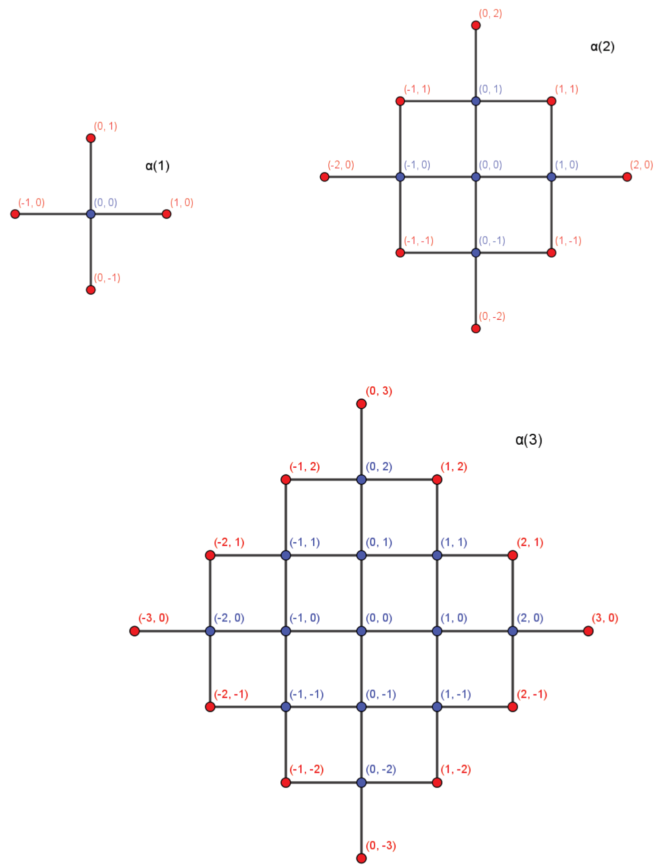

Figure 4: $\bar{B}_n\left((0,0)\right)$ for $n = 1, 2, 3$. The elements colored red are those associated with $\sigma(1)$, $\sigma(2)$, and $\sigma(3)$ respectively.

Therefore, by Theorem 9 $\alpha(n) = 1 + \sum_{i=1}^{n} 4i = 2n^2 + 2n + 1$. *Using the same technique as Example 10 we get*

$$S(z) = \left(\frac{1+z}{1-z}\right)^2$$
$$A(z) = \frac{(1+z)^2}{(1-z)^3}.$$

In deriving the above expressions for $S(z)$ *and* $A(z)$ *we need to calculate closed form expressions for various formal power series. For example, to find* $\sum_{n=1}^{\infty} nz^n$ *we note that it equals* $\sum_{n=1}^{\infty} z\frac{d}{dz}(z^n)$ *and use the usual sum of geometric series as in Example 10. For this and similar techniques see [6].*

Example 14. *Consider the infinite dihedral group* $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b : a^2, b^2 \rangle$. *We observe that* $\sigma(n) = 2$ *for* $n = 2, 3, \ldots$ *(the only two elements of length* n *are* $abab\cdots$ *and* $baba\cdots$ *where the words have* n *letters, as* a *and* b *are their own inverses), and hence, using Theorem 9,* $\alpha(n) = 2n + 1$ *for* $n = 2, 3, \ldots$. *Also, we have*

$$S(z) = \frac{1+z}{1-z}$$
$$A(z) = \frac{1+z}{(1-z)^2}.$$

Note that D_∞ has the same growth as $\mathbb{Z}$, showing that non-isomorphic groups can have the same growth. Now consider $\mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b : a^2, b^3 \rangle$. We get the following growth functions and growth series

$$\sigma(n) = \begin{cases} 1, & n = 0 \\ 3 \cdot 2^{\frac{n-1}{2}}, & n \text{ is odd} \\ 2^{\frac{n}{2}+1}, & n \text{ is even} \end{cases}$$
$$\alpha(n) = \begin{cases} 1, & n = 0 \\ 7 \cdot 2^{n-1} - 6, & n \text{ is even} \\ 5 \cdot 2^{n-1} - 6, & n \text{ is odd} \end{cases}$$
$$S(z) = \frac{(1+z)(1+2z)}{1-2z^2}$$
$$A(z) = \frac{(1+z)(1+2z)}{(1-2z^2)(1-z)}.$$

To see the $\sigma(n)$ above, observe that $a^{-1} = a$ and $b^{-1} = b^2$. Then, the elements of length n are $abab\cdots$ and $baba\cdots$ with some (or all) of the occurrences of b replaced by b^{-1}. Now when n is even, we have $abab\cdots ab$ and $baba\cdots ba$. In both words we have $\frac{n}{2}$ occurrences of b, thus there are $2^{\frac{n}{2}} + 2^{\frac{n}{2}} = 2^{\frac{n}{2}+1}$ words of length n. Now if n is odd, we have $abab\cdots a$ and $baba\cdots b$. In the first word we have $\frac{n-1}{2}$ occurrences of b, hence we have $2^{\frac{n-1}{2}}$ words starting with a. For the second word we have $\frac{n+1}{2}$ occurrences of b. Hence we have $2^{\frac{n+1}{2}}$ words starting with b or b^{-1}. Hence there are $2^{\frac{n-1}{2}} + 2^{\frac{n+1}{2}} = 3 \cdot 2^{\frac{n-1}{2}}$ words overall of length n.

Though the topic of growth of groups is really interesting for infinite (finitely generated) groups, we calculate the growth of the finite cyclic groups below. Note that, as expected, the functions $\alpha(n)$ here are eventually constant.

Example 15. *Consider the group* $\mathbb{Z}_m = \langle a : a^m \rangle$. *The Cayley graphs can be thought of as regular polygons with number of sides equal to* m. *For example, the Cayley graph for* $\mathbb{Z}_3$ *looks like a regular triangle and* $\mathbb{Z}_4$ *looks like a square. The Cayley graph of* $\mathbb{Z}_m$ *can be seen in Figure 5. We leave it to the reader to check the following formulas.*

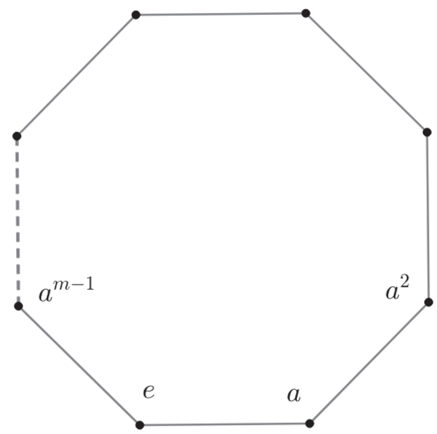

Figure 5: Cayley graph of $\mathbb{Z}_m$ with generating set $\{a\}$

If $m = 2$, then

$$\sigma(n) = \begin{cases} 1, & n = 0, 1 \\ 0, & n = 2, 3, \ldots \end{cases}$$
$$\alpha(n) = \begin{cases} 1, & n = 0 \\ 2, & n = 1, 2, \ldots \end{cases}$$
$$S(z) = 1 + z$$
$$A(z) = \frac{1+z}{1-z}.$$

If m is an even integer and greater than 2, then we have

$$\sigma(n) = \begin{cases} 1, & n = 0 \\ 2, & n = 1, 2, \ldots, \frac{m}{2} - 1 \\ 1, & n = \frac{m}{2} \\ 0, & n = \frac{m}{2} + 1, \frac{m}{2} + 2, \ldots \end{cases}$$
$$\alpha(n) = \begin{cases} 1, & n = 0 \\ 2n + 1, & n = 1, 2, \ldots, \frac{m}{2} - 1 \\ m, & n = \frac{m}{2}, \frac{m}{2} + 1, \ldots \end{cases}$$
$$S(z) = 1 + 2 \sum_{i=1}^{\frac{m}{2}-1} z^i + z^{\frac{m}{2}}$$
$$A(z) = \left(1 + 2 \sum_{i=1}^{\frac{m}{2}-1} z^i + z^{\frac{m}{2}}\right)\left(\frac{1}{1-z}\right).$$

and if m is odd and greater than 1 we have

$$\sigma(n) = \begin{cases} 1, & n = 0 \\ 2, & n = 1, 2, \ldots, \frac{m-1}{2} \\ 0, & n = \frac{m+1}{2}, \frac{m+3}{2}, \ldots \end{cases}$$

$$\alpha(n) = \begin{cases} 1, & n = 0 \\ 2n+1, & n = 1, 2, \ldots, \frac{m-1}{2} \\ m, & n = \frac{m+1}{2}, \frac{m+3}{2}, \ldots \end{cases}$$

$$S(z) = 1 + 2\sum_{i=1}^{\frac{m-1}{2}} z^i$$

$$A(z) = \left(1 + 2\sum_{i=1}^{\frac{m-1}{2}} z^i\right)\left(\frac{1}{1-z}\right).$$

Example 16. *Consider the free group on two generators $\mathbb{F}_2 = \langle a, b\rangle$. We can easily find the growth series by first finding $\sigma(n)$. We can find $\sigma(n)$ by examining the Cayley graph in Figure 6. We have $\sigma(0) = 1$ and $\sigma(1) = 4$. Notice that for each subsequent $\sigma(i)$, each point for $\sigma(i-1)$ breaks off into 3 distinct points. So $\sigma(i) = 3\sigma(i-1)$ for $i > 1$. One can see that*

$$\sigma(n) = \begin{cases} 1, & n = 0 \\ 4 \cdot 3^{n-1}, & n = 1, 2, \ldots \end{cases}.$$

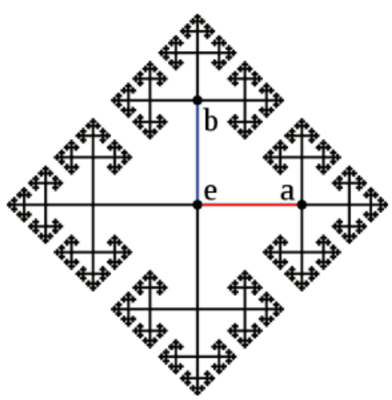

Figure 6: Cayley graph of $\mathbb{F}_2$ with generating set $\{a, b\}$

And now to find $\alpha(n)$ we use Theorem 9 to get

$$\alpha(n) = 2 \cdot 3^n - 1.$$

Now we can find the spherical growth series

$$S(z) = \sum_{n=0}^{\infty} \sigma(n) z^n = \frac{1+z}{1-3z} \text{ and } A(z) = \frac{1+z}{(1-3z)(1-z)}.$$

We can generalize for $\mathbb{F}_m = \langle a_1, a_2, \ldots, a_m \rangle$. Since there are no relations, the elements in $\bar{B}_1(e) \backslash e$ are $a_1, a_2, \ldots, a_m$ and their inverses. Hence $\sigma(1) = 2m$. Consider an element in $\bar{B}_1(e) \backslash e$, say a_i. All the elements which are adjacent to a_i and are in $\bar{B}_2(e) \backslash \bar{B}_1(e)$ are $a_i a_j$ where $a_j \neq a_i^{-1}$. Hence there are $(2m-1)$ elements branching out from a_i and since we have $2m$ elements in $\bar{B}_1(e)$, we have $\sigma(2) = 2m(2m-1)$. Now, for each element in $\bar{B}_n(e) \backslash \bar{B}_{n-1}(e)$ one can create $(2m-1)$ new elements. Hence $\sigma(n) = 2m(2m-1)^{n-1}$ for $n = 1, 2, \ldots$. The growth functions and series are

$$\begin{aligned}
\sigma(n) &= \begin{cases} 1, & n = 0 \\ 2m\,(2m-1)^{n-1}, & n = 1, 2, \ldots \end{cases} \\
\alpha(n) &= \frac{m\,(2m-1)^n - 1}{m-1} \\
S(z) &= \frac{1+z}{1-(2m-1)z} \\
A(z) &= \frac{1+z}{[1-(2m-1)z]\,(1-z)}.
\end{aligned}$$

With the above example we see that the non-abelian free groups have exponential growth.

Example 17. *The Fundamental Theorem of Finitely Generated Abelian Groups states that every finitely generated abelian group is isomorphic to a group of the form*

$$\mathbb{Z}^n \oplus \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_m}$$

where the p_i's are powers of primes. Using the growth series for $\mathbb{Z}^n$ (See Section 4) and the series for the $\mathbb{Z}_{p_i}$'s (Example 15), we get the growth series using Theorem 29. We have

$$\begin{aligned}
S(z) &= \left(\frac{1+z}{1-z}\right)^n \prod_{i,\text{where } 2\nmid p_i} \left(1 + 2\left(\sum_{j=1}^{\frac{p_i-1}{2}} z^i\right)\right) \prod_{k,\text{where } 2\mid p_k} \left(1 + 2\left(\sum_{m=1}^{\frac{p_k}{2}-1} z^m\right) + z^{\frac{p_k}{2}}\right) \\
A(z) &= \frac{(1+z)^n}{(1-z)^{n+1}} \prod_{i,\text{where } 2\nmid p_i} \left(1 + 2\left(\sum_{j=1}^{\frac{p_i-1}{2}} z^i\right)\right) \prod_{k,\text{where } 2\mid p_k} \left(1 + 2\left(\sum_{m=1}^{\frac{p_k}{2}-1} z^m\right) + z^{\frac{p_k}{2}}\right).
\end{aligned}$$

We will see later that the growth of the Finitely Generated Abelian Groups, even though they appear rather complicated, are primarily influenced by the $\mathbb{Z}^n$ component of the group. See Example 26.

Example 18. *Consider the Dihedral group, $D_n = \left\langle r, s : r^n = 1, s^2 = 1, srs = r^{-1} \right\rangle$. The Cayley graph $Cay(D_n, \{r, s\})$ can be seen in Figure 7.*

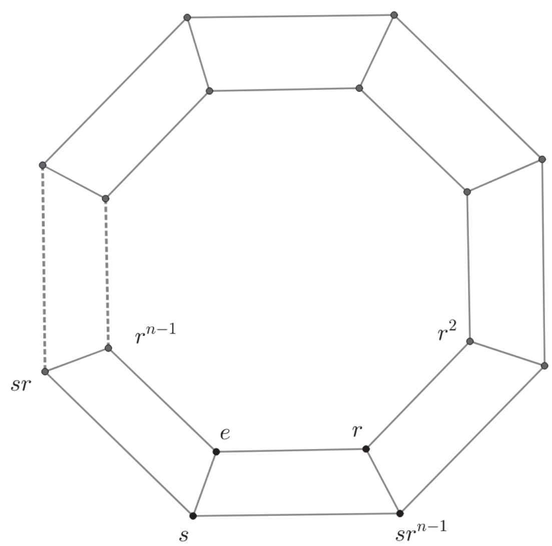

Figure 7: Cayley graph of D_n with generating set $\{r, s\}$

It is left to the reader to check the following formulas for the growth series and growth functions for $m > 2$ are

$$\sigma_{D_{2m}}(n) = \begin{cases} 1, & n = 0 \\ 3, & n = 1 \\ 4, & n = 2, 3, \ldots, m-1 \\ 3, & n = m \\ 1, & n = m+1 \\ 0, & n = m+2, m+3, \ldots \end{cases}$$

$$\alpha_{D_{2m}}(n) = \begin{cases} 1, & n = 0 \\ 4, & n = 1 \\ 4n, & n = 2, 3, \ldots, m-1 \\ 4m-1, & n = m \\ 4m, & n = m+1, m+2, \ldots \end{cases}$$

$$S_{2m}(z) = 1 + 3z + 4\sum_{i=2}^{m-1} z^i + 3z^m + z^{m+1}$$

$$A_{2m}(z) = \left(1 + 3z + 4\sum_{i=2}^{m-1} z^i + 3z^m + z^{m+1}\right)\left(\frac{1}{1-z}\right)$$

$$\sigma_{D_{2m-1}}(n) = \begin{cases} 1, & n = 0 \\ 3, & n = 1 \\ 4, & n = 2, 3, \ldots, m-1 \\ 2, & n = m \\ 0, & n = m+1, m+2, \ldots \end{cases}$$

$$\alpha_{D_{2m-1}}(n) = \begin{cases} 1, & n = 0 \\ 4, & n = 1 \\ 4n, & n = 2, 3, \ldots, m-1 \\ 4m-2, & n = m, m+1, \ldots \end{cases}$$

$$S_{D_{2m-1}}(z) = 1 + 3z + 4\sum_{i=2}^{m-1} z^i + 2z^m$$

$$A_{D_{2m-1}}(z) = \left(1 + 3z + 4\sum_{i=2}^{m-1} z^i + 2z^m\right)\left(\frac{1}{1-z}\right).$$

4 General Results

In this section we discuss some general properties of the measures of growth we introduced earlier. General references to the material in this section are [3], [5].

Proposition 19. *Consider a group generated by m elements. We have the following bounds for the two growth functions.*

1. $\sigma(n) \leq 2m(2m-1)^{n-1}$
2. $\alpha(n) \leq \frac{m(2m-1)^n - 1}{m-1}$

Proof. Both results follow from the growth functions of the free groups from Example 16. □

In the last section we calculated the exact growth functions of several groups. We will see now that as far as the asymptotic properties of the group are concerned, the exact growth functions are not so important as a certain equivalence class of those functions which we define below.

Definition 20. Let $f, g : \mathbb{N} \to \mathbb{N}$. We say $f \preceq g$ if and only if there exist $C, D > 0$ such that $f(x) \leq Cg(Dx)$. We say $f \asymp g$ if $f \preceq g$ and $g \preceq f$.

Remark 21. *In some of the literature (see [5]), the definition of the equivalence class of growth functions involves additive constants (on top the multiplicative constants C, D mentioned above). As we are interested in an essentially asymptotic notion of equivalence, our definition is sufficient.*

Proposition 22. *Let $n, m \in \mathbb{N}$.*

1. *If $n \leq m$, then $x^n \preceq x^m$.*

2. If $n \neq m$, then $x^n \not\asymp x^m$.

3. $x^n \prec 2^x$.

Proof. (Part 1) Since $x^n, x^m : \mathbb{N} \to \mathbb{N}$, for $x \geq 1$, we have $x^n \leq x^m$. Hence $x^n \preceq x^m$.

(Part 2) Without loss of generality, let $n < m$. By part 1, we know $x^n \preceq x^m$. Suppose $x^m \preceq x^n$. Then there exist $C, D > 0$ such that $x^m \leq C\,(Dx)^n$. Then

$$x^m \leq CD^n x^n \leq Bx^n$$

where $B = CD^n$. Since $x^n > 0$ for all $x \in \mathbb{N}$, we have $x^{m-n} \leq B$. Then, as $x \to \infty$, $B \to \infty$. But B must be a finite constant. Hence $x^m \not\asymp x^n$.

(Part 3) The power series expansion of 2^x is

$$\begin{aligned} 2^x &= \sum_{k=0}^{\infty} (\ln(2))^k \frac{x^k}{k!} \\ &= (\ln(2))^n \frac{x^n}{n!} + \sum_{k \neq n} (\ln(2))^k \frac{x^k}{k!}. \end{aligned}$$

Since both these summations are positive, then $2^x > (\ln(2))^n \frac{x^n}{n!}$. Hence $x^n < \left(\frac{n!}{(\ln(2))^n}\right) 2^x$ for all $x, n \in \mathbb{N}$. Also, it is easy to see that there are no $C, D > 0$ such that $2^x \leq CD^n x^n$ for all $x, n \in \mathbb{N}$. Hence, $x^n \prec 2^x$. □

For example, we see that the growth functions $\alpha(n)$ for $\mathbb{Z}$ with standard generators, $\mathbb{Z}$ with generating set $\{2, 3\}$, and $\mathbb{Z}$ with generating set $\{p, q\}$ are all equivalent (they are all eventually linear - see Examples 10 and 11). We can now make a general statement about the growth functions for the same group with distinct finite generating sets.

Theorem 23. *Let G be a group with two distinct, finite generating sets S_1, S_2 and let α_{G,S_1} and α_{G,S_2} be the corresponding growth functions. Then $\alpha_{G,S_1} \asymp \alpha_{G,S_2}$.*

Proof. Let $S_1 = \{a_1, a_2, \ldots, a_\ell\}$ and $S_2 = \{b_1, b_2, \ldots, b_m\}$. Then from Definition 4, we see that $\alpha(n) = |A|$, where $A = \{w \in G : d(e, w) \leq n\}$. Let $p = \max\{d_{S_1}(e, b_i)\}$. We can write any $w \in A$ as

$$w = c_1 c_2 \cdots c_k,\ \ c_i \in S_1, k = d_{S_1}(e, w).$$

Now since $c_i \in G$, we can write them as

$$c_i = d_{i1} d_{i2} \cdots d_{iq},\ \ d_{ij} \in S_2 \cup \{e\}.$$

Then, $d_{S_2}(e, w) \leq pk = pd_{S_1}(e, w)$. Then, $\alpha_{G,S_2}(n) \leq p\alpha_{G,S_1}(n)$, which shows that $\alpha_{G,S_2} \preceq \alpha_{G,S_1}$. A similar argument shows that $\alpha_{G,S_1} \preceq \alpha_{G,S_2}$. □

Example 24. *Recall in Example 10 and in Example 11 the growth functions α for $\mathbb{Z}$ with standard generating set and $\{2, 3\}$ generating set are both eventually linear. This is expected from the previous theorem.*

Based on Theorem 23, we make the following definition, noting that $\asymp$ is an equivalence relation.

Definition 25. Given a finitely generated group, we define the equivalence class $[\alpha]$ of its growth function α and call it the growth type of G.

Example 26.

1. *A group is finite if and only if it has growth type* $[x \mapsto 1]$.
2. $\mathbb{Z}^n$ *has growth type* $[x \mapsto x^n]$.
3. $\mathbb{F}_m$ *has growth type* $[x \mapsto 2^x]$.
4. *Finitely generated abelian groups* $\mathbb{Z}^n \oplus \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_m}$ *have growth type* $[x \mapsto x^n]$.

Moreover, we can show the following result.

Theorem 27. *Let* (G_1, S_1) *and* (G_2, S_2) *be finitely generated groups. If* $f : G_1 \to G_2$ *is a quasi-isometric embedding, then* $\alpha_{G_1,S_1} \preceq \alpha_{G_2,S_2}$.

Proof. The argument is essentially the same as that of Theorem 23. □

Example 28. *(Quasi-Isometric Rigidity of* $\mathbb{Z}^n$*) The above shows that* $\mathbb{Z}^m \sim_{q.i.} \mathbb{Z}^n$ *if and only if* $n = m$.

Now we provide two results that are useful in calculating growth series of familiar group theoretical constructions.

Theorem 29. *Let* G_1, G_2 *be finitely generated groups with corresponding finite generating sets* S_1, S_2. *Let* $G = G_1 \oplus G_2$ *and* $S = \{(s_i, 0) : s_i \in S_1\} \cup \{(0, s_j) : s_j \in S_2\}$. *Then*

$$S_{G,S}(z) = S_{G_1,S_1}(z) \cdot S_{G_2,S_2}(z).$$

Proof. The result follows from the equality $\sigma(n) = \sum_{i=0}^{n} \sigma_1(i)\sigma_2(n-i)$ (see Figure 8). □

The above theorem generalizes Example 13 and gives the following series for $\mathbb{Z}^n$:

$$S(z) = \left(\frac{1+z}{1-z}\right)^n$$
$$A(z) = \frac{(1+z)^n}{(1-z)^{n+1}}.$$

Theorem 30. *Let* G_1, G_2 *be finitely generated groups with corresponding finite generating sets* S_1, S_2. *Let* $G = G_1 * G_2$ *and* $S = S_1 \sqcup_{\text{disjoint}} S_2$. *Then*

$$S_{G,S}(z) = \frac{S_{G_1,S_1}(z) \cdot S_{G_2,S_2}(z)}{S_{G_1,S_1}(z) + S_{G_2,S_2}(z) - S_{G_1,S_1}(z) \cdot S_{G_2,S_2}(z)}.$$

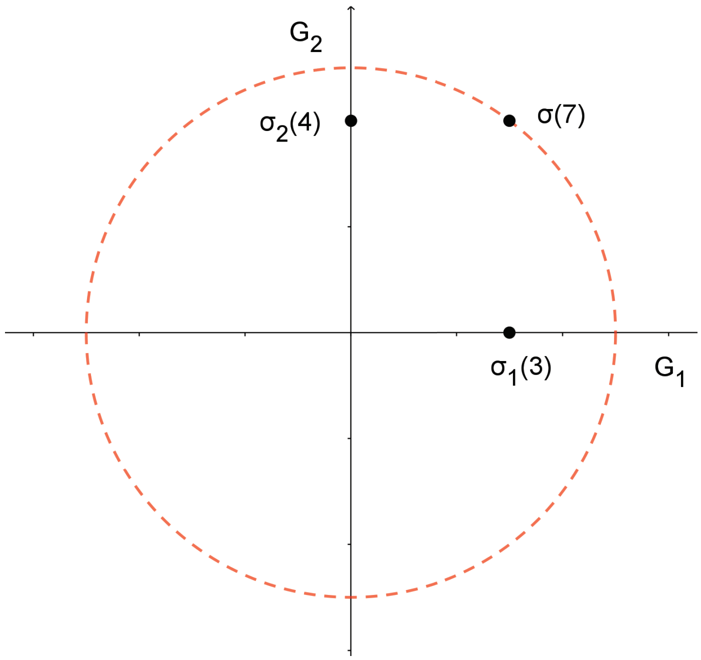

Figure 8: Cayley graph of $G_1 \oplus G_2$ with generating set S

Proof. Any element of $G_1 * G_2$ can be written in the form $gh_1g_1h_2g_2\cdots h_ng_nh$ where $g \in G_1$, $h \in G_2$, $g_i \in G_1 \setminus \{e\}$, and $h_i \in G_2 \setminus \{e\}$. Let $n \in \mathbb{N}$. Let $w = gh_1g_1h_2g_2\cdots h_ng_nh$ as above. First note $d_G(e,w) = d_{G_1}(e,g) + \sum_{i=1}^n d_{G_2}(e,h_i) + \sum_{i=1}^n d_{G_1}(e,g_i) + d_{G_2}(e,h)$. Then the number of elements of the form of w with $d_G(e,w) = k$ is

$$\sum_{\substack{i_1+i_2+\cdots+i_{n+2}=k \\ i_2,\ldots,i_{n+1}\geq 1}} \sigma_1(i_1)\sigma_2(i_2)\cdots\sigma_1(i_{n+1})\sigma_2(i_{n+2})$$

which is the kth coefficient of $S_{G_1,S_1}(z)((S_{G_1,S_1}(z)-1)(S_{G_2,S_2}(z)-1))^n S_{G_2,S_2}(z)$.

This gives us the series $S_{G_1,S_1}(z)\left((S_{G_1,S_1}(z)-1)(S_{G_2,S_2}(z)-1)\right)^n S_{G_2,S_2}(z)$ for a fixed value of n. Therefore (using the formula for the sum of a “formal” geometric series),

$$\begin{aligned} S_{G,S}(z) &= \sum_{n=0}^{\infty} S_{G_1,S_1}(z)\left((S_{G_1,S_1}(z)-1)(S_{G_2,S_2}(z)-1)\right)^n S_{G_2,S_2}(z) \\ &= \frac{S_{G_1,S_1}(z)\cdot S_{G_2,S_2}(z)}{S_{G_1,S_1}(z)+S_{G_2,S_2}(z)-S_{G_1,S_1}(z)\cdot S_{G_2,S_2}(z)}. \end{aligned}$$ □

The above theorem generalizes Example 14.

Proposition 31. *Let G be a finitely generated group and H a finitely generated subgroup. Then $\alpha_H(n) \leq \alpha_G(n)$.*

Proof. Let G be a finitely generated group with finite generating set S_1 and let H be a subgroup of G with finite generating set S_2. Define $S' = S_1 \cup S_2$, and note that S' generates G. Now consider $\alpha_{H,S_2}(n)$. Then,

$$\begin{aligned}\alpha_{H,S_2}(n) &= \#\{w \in H : d_{H,S_2}(e,w) \le n\} \\ &\le \#\{w \in G : d_{G,S'}(e,w) \le n\} \\ &= \alpha_{G,S'}(n).\end{aligned}$$

Hence, $\alpha_{H,S_2}(n) \le \alpha_{G,S'}(n)$, which means $\alpha_{H,S_2}(n) \preceq \alpha_{G,S'}(n)$. Since G has two finite generating sets S_1 and S', then by Theorem 23, $\alpha_{G,S'} \asymp \alpha_{G,S_1}$. Hence, $\alpha_{H,S_2}(n) \preceq \alpha_{G,S_1}(n)$. □

Remark 32. *The above proposition gives us a way of putting a lower bound on the growth function of a group when the growth function of a subgroup is known. For example, if a group has a non-abelian free group as a subgroup, then its growth function must be exponential.*

Example 33 (Distinct groups with the same growth function)**.** *Given any positive integer N, we demonstrate N non-isomorphic groups with the same growth function.*

We have seen that $\mathbb{Z}_2 * \mathbb{Z}_2$ and $\mathbb{Z}$ have the same growth function, but these two groups are not isomorphic. Consider the free product of $\mathbb{Z}_2$ with itself $2(N-1)$ times. We have

$$\underbrace{\mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2}_{2(N-1)\text{ times}}.$$

We know that this group has the same growth function as each of

$$\underbrace{\mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2}_{2(N-2)\text{ times}} * \mathbb{Z}$$

$$\underbrace{\mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2}_{2(N-3)\text{ times}} * \mathbb{Z} * \mathbb{Z}$$

$$\vdots$$

$$\mathbb{Z}_2 * \mathbb{Z}_2 * \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{N-2\text{ times}}$$

$$\underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{N-1\text{ times}}.$$

Hence, we have created N non-isomorphic groups with the same growth function.

The above examples bring up the following question.

Question: Can there be infinitely many groups with the same growth function?

5 Rationality of Growth Series

Groups of rational growth (groups whose growth series with every finite generating set are rational functions) have been much studied. Examples of such classes of groups are virtually abelian groups, hyperbolic groups, and the Heisenberg group (For a brief history leading to the rational growth of the Heisenberg group, see [2]). In this section we give a self-contained characterization of the rationality of the growth series (of a group with a specific generating set) in terms of recurrence relations satisfied by its growth functions. This characterization of rationality of a formal power series is interesting even outside the context of growth of groups.

A linear recurrence relation of depth M is a sequence of numbers $\{a_n\}$ where $a_{n+1} = \alpha_1 a_n + \alpha_2 a_{n-1} + \cdots + \alpha_M a_{n-M+1}$, for some fixed numbers α_i, $i = 1, \ldots, M$.

Theorem 34. *The growth series of a group (under a specific generating set) is rational if and only if its corresponding growth function eventually satisfies a linear recurrence relation of finite depth.*

Proof. ($\Leftarrow$) Suppose there are $m, N \in \mathbb{N}$ so that for all $n \geq N$

$$\sigma(n) = c_1\sigma(n-1) + c_2\sigma(n-2) + \cdots + c_m\sigma(n-m),$$

where $c_i \in \mathbb{Z}$. Then,

$$S(z) = \sigma(0) + \sigma(1)z + \sigma(2)z^2 + \cdots + \sigma(n)z^n + \cdots$$

$$\begin{aligned}
c_1 z S(z) &= c_1\sigma(0)z + c_1\sigma(1)z^2 + c_1\sigma(2)z^3 + \cdots + c_1\sigma(n-1)z^n + \cdots \\
c_2 z^2 S(z) &= c_2\sigma(0)z^2 + c_2\sigma(1)z^3 + c_2\sigma(2)z^4 + \cdots + c_2\sigma(n-2)z^n + \cdots \\
&\vdots \\
c_m z^m S(z) &= c_m\sigma(0)z^m + c_m\sigma(1)z^{m+1} + c_m\sigma(2)z^{m+2} + \cdots + c_m\sigma(n-m)z^n + \cdots.
\end{aligned}$$

Then,

$$\begin{aligned}
&S(z)\left(1 - c_1 z - c_2 z^2 - \cdots - c_m z^m\right) \\
&\quad = \sigma(0) + [\sigma(1) - c_1\sigma(0)]\, z + [\sigma(2) - c_1\sigma(1) - c_2\sigma(0)]\, z^2 \\
&\quad + \cdots + [\sigma(m) - c_1\sigma(m-1) - \cdots - c_m\sigma(0)]\, z^m \\
&\quad + \cdots + [\sigma(N-1) - c_1\sigma(N-2) - \cdots - c_m\sigma(N-m-1)]\, z^{N-1} \\
&\quad + \sum_{n=N}^{\infty} (\sigma(n) - c_1\sigma(n-1) - c_2\sigma(n-2) - \cdots - c_m\sigma(n-m))\, z^n.
\end{aligned}$$

Since $\sigma(n) = c_1\sigma(n-1) + c_2\sigma(n-2) + \cdots + c_m\sigma(n-m)$ for all $n \geq N$, the summation equals zero. Hence, S must be a rational function since, $S(z) =$

$$\frac{\sigma(0) + (\sigma(1) - c_1\sigma(0))\, z + \cdots + [\sigma(N-1) - c_1\sigma(N-2) - \cdots - c_m\sigma(N-m-1)]\, z^{N-1}}{1 - c_1 z - c_2 z^2 - \cdots - c_m z^m}.$$

($\Rightarrow$) Suppose $S(z) = \dfrac{P(z)}{Q(z)}$ is a rational function, where P and Q are polynomials and $S(z)$ is in reduced form. Let $Q(z) = a_0 + a_1 z + \cdots + a_N z^N$ and recall that $S(z) = \sum_{n=0}^{\infty} \sigma(n) z^n$. If $Q(0) = 0$, let a_m be the first non-zero coefficient in Q. Defining $Q_1(z) = z^{-m} Q(z)$ we note that $Q_1(0) \neq 0$. Then, as $Q_1(z) \sum_{n=m}^{\infty} \sigma(n-m) z^n = P(z)$, we note that replacing Q by Q_1 does not change the recurrence relation condition, which is merely shifted. If $\deg P \geq \deg Q_1$, by long division we have $\frac{P}{Q_1} = R + \frac{P_1}{Q_1}$, where R and P_1 are polynomials. Again, note that replacing $\frac{P}{Q_1}$ by $\frac{P_1}{Q_1}$ does not affect the recurrence relation condition (it is just shifted). Finally, we can assume (by re-scaling) that $Q(0) = 1$.

Summarizing, by the above discussion, we can assume $\sum_{n=0}^{\infty} \sigma(n) z^n = \frac{P(z)}{Q(z)}$ where $Q(0) = 1$ and $\deg P < \deg Q$. Let $Q(z) = 1 + a_1 z + \cdots + a_N z^N$ and let $P(z) = b_0 + b_1 z + \cdots + b_M z^M$ where $M < N$. Then we have $Q(z) \sum_{n=0}^{\infty} \sigma(n) z^n = P(z)$. Now, by equating coefficients we have,

$$\begin{gathered}
\sigma(0) = b_0 \\
\sigma(1) + a_1\sigma(0) = b_1 \Rightarrow \sigma(1) = b_1 - a_1\sigma(0) \\
\sigma(2) + a_1\sigma(1) + a_2\sigma(0) = b_2 \Rightarrow \sigma(2) = b_2 - a_1\sigma(1) - a_2\sigma(0) \\
\vdots \\
\sigma(M) + a_1\sigma(M-1) + a_2\sigma(M-2) + \cdots + a_{M-1}\sigma(1) + a_M\sigma(0) = b_M.
\end{gathered}$$

Hence $\sigma(M) = b_M - a_1\sigma(M-1) - a_2\sigma(M-2) - \cdots - a_{M-1}\sigma(1) - a_M\sigma(0)$. Then, for $n \geq M+1$

$$\sigma(n) = -a_1\sigma(n-1) - a_2\sigma(n-2) - \cdots - a_{M-1}\sigma(n-M+1) - a_M\sigma(n-M).$$

Hence, σ eventually satisfies a linear recurrence relation of depth M. □

Remark 35. *In Example 16 we noted that $\sigma(n)$ satisfies a recurrence relation and hence by the above theorem the growth series must be rational. In the same example we actually calculated the growth series and it is indeed rational.*

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