Negatively Curved Groups

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The notion of a negatively curved group is at first highly non-intuitive because it links two areas of mathematics that are not usually associated with one another. Curvature is generally a property that we associate with geometric objects like curves or surfaces in $\mathbb{R}^3$, while a group is an algebraic structure that we associate with objects like integers or matrices. However, there is a way to define a group structure on paths in a geometric object like a manifold, the so-called fundamental group of the manifold, such that certain aspects of negative curvature are reflected in the group.

Negatively curved groups are interesting not only because of their algebraic properties but also because of their applications in both computer science and art. They make up the vast majority of all fundamental groups of three-dimensional manifolds, which are spaces that look locally like the three-dimensional world we live in. A famous example of the use of negatively curved reflection groups in art is M.C. Escher's woodcut *Circle Limit IV* (1960), which illustrates the overall structure of such a group. The reason that these groups are important to computer scientists is that they are what is called "automatic" and consequently have "solvable word problem." [2]

This article is a brief exploration of negatively curved groups. In order to connect geometry to group theory we begin by describing the procedure of forming a fundamental group. Next, we review the concept of curvature for two-dimensional manifolds and develop an intuitive notion of what a negatively curved group should look like. We then turn this intuitive notion into an exact criterion that distinguishes negatively curved groups in general. Finally, we analyze three examples.

Fundamental groups

Definition. Let X be a path-connected space, that is, a space in which any two points are connected by a path in X. Two paths in X, which begin and end at

the same point, are called *homotopically equivalent* if they can be continuously deformed into one another while keeping their endpoints fixed. If we denote the concatenation of two paths by "$*$" (first running along one path and then along the other path), the set G of all equivalence classes of paths, which begin and end at some fixed base point p of X, forms a group under the operation "$*$". We call $(G, *)$ the *fundamental group* of X.

Consider, for example, the fundamental group $(G, *)$ of a torus, that is, the surface of a doughnut. Every loop can be deformed into a product of loops that wind around the top and of loops that wind around the doughnut hole.
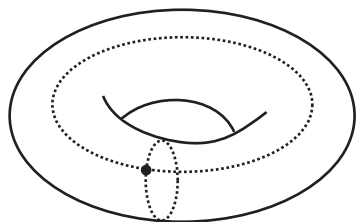
Denoting these two types of generating loops by a and b, respectively, one can check that $a * b = b * a$ in G. It follows that $(G, *)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

In order to get some idea as to what a negatively curved fundamental group might look like, we now consider this construction in the context of general surfaces and their curvature. Exact definitions of these two concepts can be found in [5]. Here, we will only give a brief review of the curvature of surfaces in 3-space.
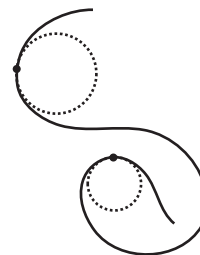
Curvature

Given a point p on a surface M, we would like to measure the curvature of M at p. In order to define the notion of curvature for surfaces, we recall the corresponding notion for planar curves.

The *curvature* of a circle of radius r is defined to be $\frac{1}{r}$. (The larger the circle, the smaller its curvature.) Given a planar curve C, one way of describing its curvature at a point p is to look at the *osculating circle* at p, which is the circle that best approximates the curve at p. It can be found as follows: take three points on the curve, call them a, b, and c. These three points uniquely determine a circle. The osculating circle will be the circle formed by letting a, b, and $c$ limit to the point p, along the curve. The curvature κ of C at p is defined to be $\frac{1}{r}$, where r is the radius of the osculating circle at p. If the planar curve C is a line, then the osculating circle has infinite radius and thus the curvature is equal to zero, which is consistent with the intuitive notion of curvature.

Now, in order to measure the curvature of a surface M at a point p, we consider certain lines passing through p and examine their curvature. Specifically, if we are given a surface M in $\mathbb{R}^3$, we can find a normal vector to the surface at the point p. If we look at all possible planes that contain the normal vector and intersect these planes with the surface we will get a collection of planar curves on the surface that pass through p. If we compute the curvature of all of these curves in their respective planes using the osculating circle (recording the curvature as positive or negative depending on whether the osculating

circle opens in the direction of the normal vector or not), then the *Gaussian curvature* of M at p is the product $\kappa_1 \cdot \kappa_2$, where κ_1 is the minimum of the curvatures of all those curves and κ_2 is their maximum.

The shape of a surface M near a point p of negative Gaussian curvature resembles a saddle. If the Gaussian curvature of M at p is positive then, locally, it looks like a dome. If, on the other hand, the Gaussian curvature is zero at p, then the surface M, near p, either looks flat like a table top (at least in its quadratic approximation) or like a cylinder (bending in only one direction).
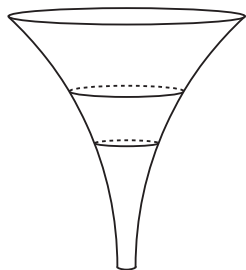
One way to visualize the curvature of a surface at a point is to make the surface shiny and to consider the reflection that can be seen near that point. If the reflection reverses both left and right, and up and down (or neither) then the surface is positively curved. An example of this would be a spoon, where the point is located at the bottom of the bowl of the spoon. If the reflection only reverses one direction, then the surface is negatively curved at that point. This can be observed, for example, at a point on the bell of a tuba, somewhat like the figure on the left.

It is a fact that a closed (compact) surface, which has negative Gaussian curvature at every one of its points, cannot be formed in $\mathbb{R}^3$ but requires at least $\mathbb{R}^4$. We think of the fundamental groups of such negatively curved closed surfaces, or negatively curved closed manifolds in general, as the prototypical examples of negatively curved groups. Triangles drawn on negatively curved surfaces (or manifolds) are thinner than usual: their sides tend to bend inward. Based on this principle of thin triangles, we would like to find a more general criterion for when a group should be called negatively curved. To this end, we will need to construct a geometric object called the Cayley graph. To understand the construction, we first develop some tools.

Cayley graphs and their geometry

A *word*, in a given set $S = \{a_i \mid i \in I\}$ of symbols, is a finite sequence of symbols a_i and their formal inverses a_i^{-1}. For convenience, we will denote the empty word as e. We call a word *reduced* if no symbol is adjacent to its inverse. The set of all reduced words under the operation $*$ of concatenation, followed by reduction (repeatedly eliminating adjacent pairs of inverses), forms a group, called the *free group F* on the generators S. If we are also given a set R of relationships among the generators, like $a_4 a_1^{-1} a_3 = e$, we can construct the largest group generated by S, in which all of these relations hold, by forming the quotient $G = F/N$, where N is the smallest normal subgroup of F which contains R. We call $\langle S \mid R \rangle$ a presentation for the group G. For example, the fundamental group of the torus above has the finite presentation $\langle a, b \mid aba^{-1}b^{-1} = e \rangle$. All our group presentations will be finite.

Given a group presentation $G = \langle S \mid R \rangle$, we can create a directed graph, called the *Cayley graph*, in the following manner. There is one vertex for each element of the group. If $y = xa_i$ with $x, y \in G$ and $a_i \in S$, then there

is a directed edge from x to y labelled a_i. The Cayley graph of the above presentation of the fundamental group of the torus, for example, looks like an integer grid in $\mathbb{R}^2$, as depicted on the next page.

If we declare each edge of the Cayley graph to have length equal to 1 unit, we can define the distance between two points as the length of a shortest path connecting them. Such a shortest (continuous) path is also called a *geodesic*. The Cayley graph is therefore an example of what is called a *geometry*: a metric space in which closed metric balls are compact and any two points are connected by a geodesic.

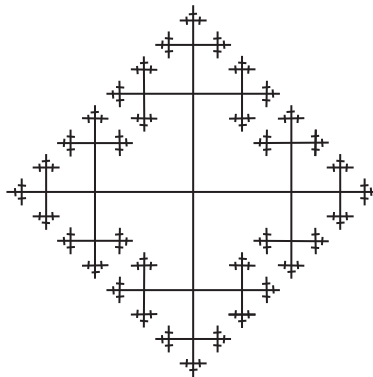We are now ready to formulate our curvature condition for groups.

Definition. The group $G = \langle S \mid R \rangle$ is called *negatively curved* if its Cayley graph has uniformly thin triangles in the following sense: there is a constant $\delta \geqslant 0$ such that each point on the side s_1 of a geodesic triangle $s_1 s_2 s_3$ lies within δ of some point of the union $s_2 \cup s_3$ of the other two sides.

It can be shown that this definition is independent of the specific presentation of the group G (see [3]) and that, as expected, the fundamental groups of negatively curved manifolds are negatively curved groups.

Examples

Now it would be very nice to look at some examples of this definition and figure out whether certain groups are negatively curved or not.

The first example that we will look at is the free group on two generators. The first thing that we need to do is construct the Cayley graph for the presentation $\langle a, b \mid \text{no relations} \rangle$. Every vertex x of the Cayley graph has four neighbors, namely xa, xb, $xb^{-1}$ and xa^{-1}. The graph is shown on the right. We observe that a path in this particular Cayley graph is a geodesic if and only if it does not backtrack, and that there is a unique geodesic between any two points. Thus, if we look at any geodesic triangle with sides s_1, s_2, and s_3, we find that $s_1 \subseteq s_2 \cup s_3$. We conclude that all geodesic triangles are δ-thin with $\delta = 0$. Therefore, the free group on two generators is negatively curved.

The second group that we will look at is the fundamental group of the torus: $\langle a, b \mid aba^{-1}b^{-1} = e \rangle$. The Cayley graph for this group presentation is the familiar integer grid in $\mathbb{R}^2$ shown on the next page. The torus cannot be deformed so as to have negative curvature at every one of its points. We therefore do not expect this group to be negatively curved. It is important to note that geodesics are not unique in this graph. For example, you can go from e at $(0,0)$ to $a^2 b^2$ at $(2,2)$ by six different geodesic paths: $aabb$, $abab$, $baab$, $abba$, $baba$, and $bbaa$. In general, every geodesic path is a stairstep path so that there is a total of $\binom{|m|+|n|}{|n|}$ geodesic paths from e to $a^m b^n$.
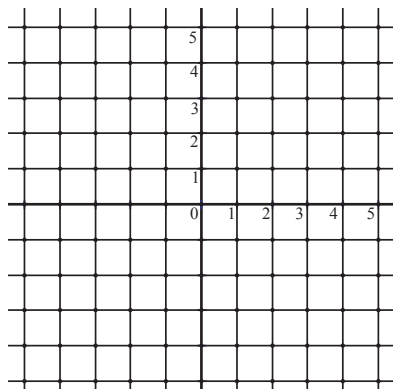
In order to show that there is no $\delta \geqslant 0$ such that the geodesic triangles in this Cayley graph are uniformly δ-thin, consider the following geodesic triangle: let s_1 be the straight line path connecting $(2\delta, 2\delta)$ and $(2\delta, -2\delta)$, let s_2 be the zigzag path from the origin e to $(2\delta, 2\delta)$ which begins in the y-direction, and let s_3 be a similarly defined path from e to the point $(2\delta, -2\delta)$. Consider the point $P(0, 2\delta)$ on s_1. Then the distance from P to any point on the sides s_2 or s_3 is at least 2δ. Hence, this group is not negatively curved.

The final example that we will look at is the orientation preserving subgroup of the group generated by the reflections R_1, R_2, and R_3 through the sides of a triangle in the hyperbolic plane with opposed angles of $\frac{\pi}{3}$, $\frac{\pi}{7}$ and $\frac{\pi}{2}$, respectively. The resulting tiling is depicted on the right, in the Poincaré disk model. This group has presentation $\langle a, b \mid a^2 = e, b^3 = e, (ab)^7 = e \rangle$, where a and b are the rotations $a = R_1 \circ R_2$ and $b = R_2 \circ R_3$. A part of the Cay-

ley graph for this group is pictured below, where the dotted lines represent multiplication by b and the solid lines represent multiplication by a. (Since $a = a^{-1}$, every outgoing edge for a has been combined with the corresponding incoming edge for a^{-1} into one single solid edge.) A geodesic edge path in this graph is a reduced word that does not contain instances of words like bb, $(ab)^4$ or $(ab)^3 ab^{-1}$, which go more than half way around a 3-gon, a 14-gon or their joined 15-gon, because these could be replaced by b^{-1}, $(b^{-1}a^{-1})^3$ or $(b^{-1}a^{-1})^3 b$, respectively, shortening the path.
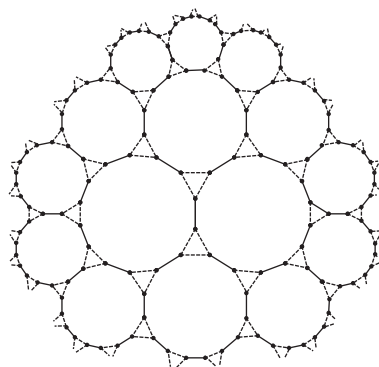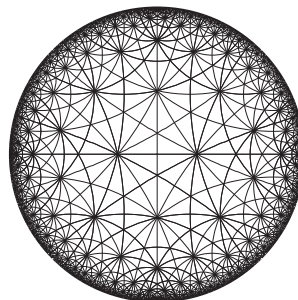
We would like to show that this group is negatively curved. In order to check if the Cayley graph has uniformly thin triangles, consider two geodesic edge paths, represented by words A and B, such that A connects the group element x to the group element y, and B connects y to z. It is possible for AB to not be a geodesic path, because words that were not previously allowed could have been introduced into AB, where the end of A is connected to the beginning of B. First, reduce the concatenation AB by cancelling generators with

their inverses. Next, look for a word of the type mentioned above, which goes more than half way around a 3-gon, 14-gon or 15-gon, and replace it by its shortcut the other way around the same 3-gon, 14-gon or 15-gon, respectively. Note that we may have to repeat the two steps of this process several times, before we finally arrive at a geodesic edge path C connecting x to z. However, the number of such shortcut replacements is bounded by the length of the word AB. Therefore, if we apply this shortening algorithm to the perimeter word of a geodesic triangle, the number of necessary replacements is bounded by the length of the perimeter. Since the perimeter spells out a trivial word in the group, these cyclic replacements provide a sense of area for this triangle in terms of 3-gons, 14-gons and 15-gons, which is bounded by a linear function of the perimeter. As one might expect, such a uniform linear bound of area, in terms of perimeter, forces triangles to be uniformly thin. (For details see [1, pp. 419–421].) We conclude that this group is negatively curved.

Conclusion

Not only is every fundamental group of a closed negatively curved manifold negatively curved in the above sense, but many other classes of groups are negatively curved as well. Indeed, according to Gromov, almost every finitely presented group is negatively curved. Many geometric ideas and techniques have thus found profound applications to group theory. Conversely, the study of negatively curved groups can lead to important discoveries about three and four-dimensional manifolds that otherwise might be very difficult to make.

References

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