**Student Projects**

# Non-Destructive Testing of Thermal Resistances for a Single Inclusion in a 2-Dimensional Domain

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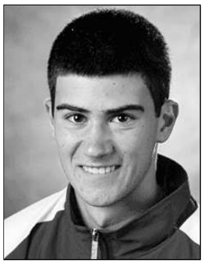

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## Introduction

The ability to determine whether there are any defects in the interior of an object without destroying it is an invaluable tool in today's industries. One popular method for doing this is steady state thermal imaging. This paper outlines a process where steady state heat flow is used to determine the constitutive law governing how a single inclusion in an object resists the flow of heat. In particular, we will be concerned with circular domain $D$ encapsulated inside an outer region $\Omega$. Our goal is to produce a function which quantifies the behavior of the heat flow across the interface between $\Omega$ and $D$. That is, as we move along the inclusions boundary, we want to know at any particular point how much heat flow is being impeded. From this, we can make inferences as to how much disbanding or corrosion has occurred on the interface between $\Omega$ and $D$.

## The Forward Problem

Let $\Omega$ be a bounded region of $\mathbb{R}^2$ with boundary $\partial\Omega$. We will assume that after appropriate scaling, $\Omega$ has thermal conductivity and diffusivity equal to one. Let $D \subset \Omega$ be an inclusion (of any shape for the moment) with presumably different thermal properties from those of $\Omega$ - say $D$ has conductivity $\alpha$ and diffusivity $\kappa$. A time independent heat flux $g$ is applied to $\partial\Omega$ for some time: we assume this time is long enough so that the temperature inside $\Omega$ stabilizes at some function $u(x, y)$. Thus the function $u$ satisfies the 2-dimensional steady state heat equation

$$\Delta u \;:=\; \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

in $\Omega \setminus \partial D$, as well as satisfies the boundary condition

$$\frac{\partial u}{\partial \mathbf{n}}(\mathbf{p}) \;=\; g(\mathbf{p}) \tag{2}$$

for $\mathbf{p} \in \partial\Omega$, where $\mathbf{n}$ is the unit outward normal vector on $\partial\Omega$. The differential operator $\Delta$ is known as the Laplacian. Functions which solve (1) on a domain are said to be harmonic there. Such functions are of great importance in several areas of mathematics and physics.

Throughout this paper, we will utilize the following notation:

$$u^+(\mathbf{p}) = \begin{cases} u(\mathbf{p}), & \text{if } \mathbf{p} \in \;\Omega \setminus \overline{D} \\ \lim\limits_{\substack{\mathbf{z} \to \mathbf{p} \\ \mathbf{z} \in \Omega \setminus \overline{D}}} u(\mathbf{z}), & \text{if } \mathbf{p} \in \partial D \end{cases}$$

and

$$u^-(\mathbf{p}) = \begin{cases} u(\mathbf{p}), & \text{if } \mathbf{p} \in \; D \\ \lim\limits_{\substack{\mathbf{z} \to \mathbf{p} \\ \mathbf{z} \in D}} u(\mathbf{z}), & \text{if } \mathbf{p} \in \partial D \end{cases}$$

If we first assume that the interface between $\Omega$ and $D$ is un-corroded, then we should have $[u](\mathbf{p}) = 0$ for any $\mathbf{p} \in \partial D$ where $[u](\mathbf{p}) = u^+(\mathbf{p}) - u^-(\mathbf{p})$ is the jump in the temperature $u$ at the point $\mathbf{p}$. That is, we expect the temperature to be continuous across $\partial D$. We should also require that the rate at which energy flows past $\mathbf{p}$ from inside $D$ equals the rate at which energy flows past $\mathbf{p}$ from outside $D$, *i.e.*, conservation of energy. This can be quantified as

$$\frac{\partial u^+}{\partial \mathbf{n}}(\mathbf{p}) = \alpha \frac{\partial u^-}{\partial \mathbf{n}}(\mathbf{p}), \;\; \mathbf{p} \in \partial D$$

where $\alpha$ is the thermal conductivity of $D$ and $\mathbf{n}$ is the unit outward normal vector on $\partial D$.

Now, suppose that $\partial D$ has corroded. We expect this would manifest itself as some type of thermal contact resistance to the flow of heat over $\partial D$ and thus would be detectible by measuring the temperature on $\partial\Omega$. In this case, we require that

$$\frac{\partial u^+}{\partial \mathbf{n}}(\mathbf{p}) = k(\mathbf{p})[u](\mathbf{p}) = \alpha\frac{\partial u^-}{\partial \mathbf{n}}(\mathbf{p}),\ \mathbf{p} \in \partial D \tag{3}$$

for some function $k \geq 0$ which quantifies the magnitude of the contact resistance across $\partial D$. The case where $k \equiv 0$ implies no heat flows over $\partial D$ and thus we have a complete disband of the interface between the two materials. Note that since we don't expect that the corrosion levels will be the same at every point on the boundary of the inclusion, $k$ is required to be a function of position.

The forward problem presented above involves solving for the steady temperature $u(x, y)$ given that $u$ satisfies the Neumann boundary value problem (1)-(3) and that the flux $g$ and $\frac{\partial u^+}{\partial \mathbf{n}}$ are known. If $\int_{\partial\Omega} g\ ds = 0$, then it is known that this problem has a unique solution up to an additive constant: to ensure a unique solution, we add the normalization condition $\int_{\partial\Omega} u\ ds = 0$ (see [1, pp. 25-46]). The restriction on $g$ follows from conservation of energy: what flows into $\Omega$ must flow out. This can be proven by assuming that $\Delta u \equiv 0$ on a region $\Gamma$ where the Divergence theorem can be applied. Then we have

$$\int_{\partial\Gamma} \frac{\partial u}{\partial \mathbf{n}}\ ds = \int_{\Gamma} \Delta u\ da = 0$$

where $ds$ represents an element of arc-length and $\mathbf{n}$ is the unit outward normal to $\partial\Gamma$. The forward problem will not, however, be the object of our analysis. We will instead be interested in the following inverse problem: given the heat flux $g$ and the solution to (1)-(3) on $\partial\Omega$, determine the function $k$ in (3). That is, from knowing the time-independent heat flux $g$ being applied to $\partial\Omega$ as well as the temperature $u$ at each point on $\partial\Omega$, determine how much the heat flow across the boundary of $D$ is being impeded.

# Preliminaries

We begin with a two-dimensional domain $\Omega$ and a single inclusion $D$ completely contained inside $\Omega$. Let us recall Green's second identity: for any bounded region $\Gamma \subset \mathbb{R}^2$, with sufficiently smooth boundary $\partial\Gamma$, if $u,\ v \in C^2(\Gamma \cup \partial\Gamma)$, then

$$\int\int_{\Gamma} (u\Delta v \ - \ v\Delta u)\ dA \ = \ \int_{\partial\Gamma} \left( u\frac{\partial v}{\partial \mathbf{n}} \ - \ v\frac{\partial u}{\partial \mathbf{n}} \right)\ ds \tag{4}$$

where again $ds$ represents an element of arc-length and $\mathbf{n}$ is the unit outward normal to $\partial\Gamma$ (see [2, pp. 525-528]). This identity is a direct consequence of the Divergence Theorem and will be the driving force for recovering the needed information to reconstruct $k$.

Due to the geometry of the problem, we will utilize polar coordinates for most of our computations. For simplicity, we will assume $\Omega$ is the unit disk. We will take the origin of our coordinate system to be the center of $\Omega$ and let $(r, \theta)$ be a point in $\Omega$ in standard polar coordinates. Let $D$ be a disk with radius $R_D$ and have its center at the point $(x_0, y_0)$. Also, let $(\rho, \beta)$ represent polar coordinates as if the center of our coordinate system was $(x_0, y_0)$. So, in our $(r, \theta)$ coordinate system, (1) and (2) become

$$\Delta u(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \quad + \quad \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \text{ in } \Omega \setminus (\partial D), \text{ and} \tag{5}$$

$$\frac{\partial u}{\partial r}(1,\theta) \quad = \quad g(1,\theta). \tag{6}$$

We also need to recall the basic definition of a Fourier series: if $f(x)$ is a piecewise-smooth, $2\pi$-periodic continuous function defined on the interval $[-\pi, \pi]$, it can be represented as a Fourier series on that interval as

$$f(x) \quad = \quad \frac{a_0(f)}{2} \quad + \sum_{n=1}^{\infty}(a_n(f)\cos(nx) \ + \ b_n(f)\sin(nx)) \tag{7}$$

where the Fourier coefficients are given by

$$a_n(f) \quad = \quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad \text{for } n \geq 0,$$

$$b_n(f) \quad = \quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx, \quad \text{for } n \geq 1.$$

Using these tools, we have everything we need to recover the function $k(\mathbf{p})$.

# Recovering k(p)

Our goal is to reconstruct the flux $\frac{\partial u^+}{\partial \mathbf{n}}$ and the jump $[u]$ in terms of Fourier series so that we can isolate the function $k$ from (3). Classically, we would be given a function and asked to represent it as a Fourier series using the above definitions. We, however, cannot do this directly since we do not know the function we are trying to reconstruct (that's the whole point!). So how can we recover the Fourier coefficients of an unknown function?

The answer is to make use of Green's second identity. Let $u(r, \theta)$ be a solution to (5) and (6) and let $v(r, \theta)$ be any function harmonic on $\Omega \setminus (\partial D)$. Applying (4) to $\Omega \setminus D$, keeping in mind that $\mathbf{n}$ represents the unit outward normal on $\Omega \setminus D$, we have

$$\int_{\partial\Omega}\left(u\frac{\partial v}{\partial \mathbf{n}} - vg\right)ds \ - \int_{\partial D}\left(u^+\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u^+}{\partial \mathbf{n}}\right)ds = 0.$$

The first integral above (over $\partial\Omega$) is often referred to as the reciprocity gap integral, and is denoted hereafter as $RG(v)$. Note that $RG$ is a linear operator and, for any function $v$ harmonic in $\Omega \backslash D$, is computable from known boundary data for $u$. This gives us the following useful identity:

$$RG(v) = \int_{\partial D} \left( u^+ \frac{\partial v}{\partial \mathbf{n}} - v \frac{\partial u^+}{\partial \mathbf{n}} \right) ds. \tag{8}$$

This identity has been used extensively in examining the problem of finding the location and constitutive law governing heat flow across a crack $\sigma$ in a domain $\Omega$ and we refer the reader to [1, pp. 25-46] for more information.

Notice that in (8), we are free to use any harmonic test function $v$ we choose. Therefore, we will use specifically designed test functions to extract the Fourier coefficients of the flux, as well as the jump. With this information in hand, we can reconstruct the flux and the jump on $\partial D$, as in (7), and thus determine the function $k$. We first start with recovering the flux in $u$ across $\partial D$.

## Recovering the Flux

As described above, we wish to design a certain family of test functions $v_n(r, \theta)$ from which we can use (8) to extract the Fourier coefficients of $\frac{\partial u^+}{\partial \mathbf{n}}$ on the interval $[-\pi, \pi]$. Such test functions would have to be harmonic in $\Omega \setminus (\partial D)$ (that is, satisfy equation (5)) in order for (8) to be valid. Note that since $D$ is a circle, the normal derivative of $v_n$ on $\partial D$ is simply the radial derivative in the $(\rho, \beta)$ coordinate system. One can easily verify by direct substitution that the following families of test functions are harmonic in $\Omega \setminus (\partial D)$ and have zero radial derivative on $\partial D$ for any non-negative integer $n$:

$$\begin{aligned} v_n^c(\rho, \beta) &= \left(\rho^n + R_D^{2n} \rho^{-n}\right) \cos(n\beta) \\ v_n^s(\rho, \beta) &= \left(\rho^n + R_D^{2n} \rho^{-n}\right) \sin(n\beta) \end{aligned}$$

where we recall that $R_D$ is the radius of $D$. Substituting $v_n^c$ into equation (8) yields (since $ds = R_D \, d\beta$)

$$RG(v_n^c) = -\int_{\partial D} 2R_D^n \frac{\partial u^+}{\partial \mathbf{n}} \cos(n\beta) ds = -2R_D^{n+1} \int_{-\pi}^{\pi} \frac{\partial u^+}{\partial \mathbf{n}} \cos(n\beta) d\beta.$$

Similar computations can be carried out for $v_n^s$. Therefore, we have that the Fourier sine and cosine coefficients of $\frac{\partial u^+}{\partial \mathbf{n}}$ are given by

$$\begin{aligned} a_n \left( \frac{\partial u^+}{\partial \mathbf{n}} \right) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\partial u^+}{\partial \mathbf{n}} \cos(n\beta) d\beta = \frac{-RG(v_n^c)}{2\pi R_D^{n+1}}, \ n \geq 0 \\ b_n \left( \frac{\partial u^+}{\partial \mathbf{n}} \right) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\partial u^+}{\partial \mathbf{n}} \sin(n\beta) d\beta = \frac{-RG(v_n^s)}{2\pi R_D^{n+1}}, \ n \geq 1 \end{aligned} \tag{9}$$

respectively. We have thus reconstructed the flux across $\partial D$ as

$$\frac{\partial u^+}{\partial \mathbf{n}}(R_D, \beta) = \frac{a_0 \left( \frac{\partial u^+}{\partial \mathbf{n}} \right)}{2} + \sum_{n=1}^{\infty} \left( a_n\left( \frac{\partial u^+}{\partial \mathbf{n}} \right) \cos(n\beta) + b_n\left( \frac{\partial u^+}{\partial \mathbf{n}} \right) \sin(n\beta) \right) \tag{10}$$

in the $(\rho, \beta)$ coordinate system. Note that from (3), we have solved for the quantity $k(R_D, \beta)[u](R_D, \beta)$. So, by determining $[u]$, we will be able to recover $k$.

## Recovering the Temperature Jump

To recover the jump, we will have to recover $u^+$ and $u^-$ separately, then take their difference (recall that $[u] = u^+ - u^-$). In order to recover $u^+$, we turn back to equation (8). Consider the family of test functions $q_n$ defined by

$$q_n^c(\rho, \beta) = \left(\rho^n - R_D^{2n}\rho^{-n}\right)\cos(n\beta), \tag{11}$$
$$q_n^s(\rho, \beta) = \left(\rho^n - R_D^{2n}\rho^{-n}\right)\sin(n\beta). \tag{12}$$

Again, one can verify that these functions are harmonic in $\Omega \setminus (\partial D)$ and are equal to zero on $\partial D$ for any non-negative integer n. By substituting (11) and (12) into (8), we see that the Fourier sine and cosine coefficients for $u^+$ are given by

$$a_n\left(u^+\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} u^+\cos(n\beta)d\beta = \frac{RG(q_n^c)}{2n\pi R_D^n},\ n \geq 1$$
$$b_n\left(u^+\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} u^+\sin(n\beta)d\beta = \frac{RG(q_n^s)}{2n\pi R_D^n},\ n \geq 1$$

respectively. Notice that we have determined each of the coefficients except for $a_0(u^+)$. We'll address this issue after we recover $u^-$. For now, we'll consider the quantity $u^+$ as known.

We now need only to recover $u^-$. We cannot use equation (8) here since $u^-$ is not present. So, we slightly adapt the above procedure by applying Green's second identity on $D$ and see that for any function $v$ harmonic in $D$,

$$\int_{\partial D} u^-\frac{\partial v}{\partial \mathbf{n}}ds = \int_{\partial D} v\frac{\partial u^-}{\partial \mathbf{n}}ds. \tag{13}$$

Define the following families of test functions

$$z_n^c(\rho, \beta) = \alpha\rho^n\cos(n\beta)$$
$$z_n^s(\rho, \beta) = \alpha\rho^n\sin(n\beta)$$

for any non-negative integer $n$ where $\alpha$ again is the thermal conductivity of $D$. Substituting $z_n^c$ for $v$ into (13) yields

$$\int_{\partial D} R_D^{n-1}\alpha n u^-\cos(n\beta)ds = \int_{\partial D} R_D^n\alpha\frac{\partial u^-}{\partial \mathbf{n}}\cos(n\beta)ds.$$

Recalling equation (3), this becomes, on noting again that $ds = R_D d\beta$,

$$\begin{aligned}\int_{\partial D} R_D^{n-1}\alpha n u^-\cos(n\beta)ds &= \int_{\partial D} R_D^n\frac{\partial u^+}{\partial \mathbf{n}}\cos(n\beta)ds \\ &= \pi R_D^{n+1}a_n\left(\frac{\partial u^+}{\partial \mathbf{n}}\right) \\ &= \frac{-RG(v_n^c)}{2}.\end{aligned}$$

The last equality is obtained using (9). A similar computation can be carried out for $z_n^s$. Therefore, the Fourier cosine and sine coefficients of $u^-$ are given by

$$a_n\left(u^-\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} u^- \cos(n\beta)d\beta = \frac{-RG(v_n^c)}{2n\pi\alpha\ R_D^n},\ n \geq 1$$
$$b_n\left(u^-\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} u^- \sin(n\beta)d\beta = \frac{-RG(v_n^s)}{2n\pi\alpha R_D^n},\ n \geq 1$$

respectively. Note, just as with $a_0(u^+)$, we have no information about $a_0(u^-)$. However, for the moment, we will consider the quantity $u^-$ as being known.

We are now in position to reconstruct $[u]$. By subtracting the coefficients of $u^+$ and $u^-$ term by term, we see that we can represent the jump as

$$[u](R_D, \beta) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos(n\beta) + B_n \sin(n\beta)\right) \tag{14}$$

with coefficients

$$A_n = \frac{RG(w_n^c)}{2n\pi R_D^n},\ n \geq 1,$$
$$B_n = \frac{RG(w_n^s)}{2n\pi R_D^n},\ n \geq 1$$

where

$$w_n^c(\rho,\beta) = q_n^c(\rho,\beta) + \frac{1}{\alpha}z_n^c(\rho,\beta) = \left[\left(\frac{1}{\alpha}+1\right)\rho^n + \left(\frac{1}{\alpha}-1\right)R_D^{2n}\rho^{-n}\right]\cos(n\beta),$$

and the function $w_n^s$ is defined similarly. We have thus recovered all of the information necessary to reconstruct $k(\beta)$, except for the constant term (since we have no information about $a_0(u^+)$ or $a_0(u^-)$).

To determine the constant term $\frac{A_0}{2}$ above, we recall that for any given $\beta$, $k(\beta) \geq 0$. In (10), we showed the complete reconstruction of the product $k(\beta)[u](\beta)$. By conservation of energy, we know that the net amount of energy entering and leaving $D$ must be the same. This fact is quantified as

$$\int_{\partial D} \frac{\partial u^+}{\partial \mathbf{n}}\, ds = 0. \tag{15}$$

This can also be seen by using $v(r,\theta) = a \in \mathbb{R}$ in (8): the normalization requirement $\int_{\partial\Omega} u\ ds = 0$, along with the condition that $\int_{\partial\Omega} g\ ds = 0$ (see the end of section 2), implies that $RG(a) = 0$ for any constant $a$. If the flux across $\partial D$ is not zero (which follows if $g$ is not identically zero), then from (15), we know that $\frac{\partial u^+}{\partial \mathbf{n}}$ must change signs at least once in the interval $[-\pi,\ \pi]$. Since $k$ is non-negative, this means that $[u]$ must change signs at the same time the flux does. It is easy to see that this uniquely determines the value of our unknown constant (compute $[u]$ via (14) with $A_0 = 0$, then adjust $A_0$ so that $[u](\beta^*) = 0$, where $\beta^*$ is any point at which $\frac{\partial u}{\partial \mathbf{n}}(\beta^*)$ changes signs). Thus, we have completely recovered the jump.

## Generalization

The requirement made by (3) was nothing more than a special case of a more general requirement (see [1, pp. 25-46]). The above procedure can be easily generalized to the case where (3) is replaced by

$$\frac{\partial u^+}{\partial \mathbf{n}}(\mathbf{p}) \;=\; F\left(\mathbf{p},\ [u](\mathbf{p})\right), \text{ for } p \in \partial D,$$

where we require that $F$ is continuous, $F(\mathbf{p}, 0) = 0$, and, when regarded as a function of the jump, is a non-decreasing odd function. The reason for this is that heat flows in proportion to the temperature difference (from higher temperatures to lower temperatures). If there is no temperature difference at the boundary of the inclusion $D$, then no heat should flow through it and thus $F(\mathbf{p}, 0) = 0$. When the jump is positive at a point $\mathbf{p} \in \partial D$, the temperature on the exterior of the inclusion is higher than the temperature on the interior of the inclusion in a small neighborhood of $\mathbf{p}$. This implies that the heat flow should be positive. So, the larger the temperature jump at $p$, the greater the heat flow should be through it. Therefore, $F$ should be required to be non-decreasing as a function of $[u]$. The reason for $F$ being odd is simple: if you have a jump $[u](\mathbf{p}_0)$ and flux $F_0$ at that point, then you would expect $F(\mathbf{p}_0, -[u](\mathbf{p}_0))$ to have the same magnitude, but the heat should flow in the opposite direction giving $F(\mathbf{p}_0, -[u](\mathbf{p}_0)) = -F(\mathbf{p}_0, [u](\mathbf{p}_0))$. Hence, $F$ should be odd with respect to the jump.

The generalization to recover the unknown function $F$ is similar to the above case where we took $F(\mathbf{p}, [u](\mathbf{p})) = k(\mathbf{p})[u](\mathbf{p})$. Since we can recover the flux and the jump on $\partial D$ using the above procedure, by evaluating each of them at an angle $\beta \in [-\pi, \pi]$, we know that

$$\frac{\partial u^+}{\partial \mathbf{n}}(R_D, \beta) \;=\; F\left(R_D, [u](\beta)\right).$$

Therefore, we can reconstruct the function $F$ on the set of all temperature jumps on $\partial D$ without any modification to the above procedure.

## Conclusions

In conclusion, we have shown how one can determine the levels of thermal resistances present on the interface of two materials with presumably different thermal properties in the general case where the flux across $\partial D$ is an arbitrary function of the jump. However, there are some limitations to our procedure. One, it relies heavily on the fact that $D$ is circular. It turns out that the problem of $D$ being nearly circular, the best linear approximation is to solve the problem on an approximating circular region $D'$ to $D$. However, if we allow $D$ to vary largely from being a circle, then our approximations give us little to no useful information about the actual problem. In addition to this, any error incurred in measuring the temperature $u$ around the boundary of $\Omega$ will be magnified in our determination of the fourier coefficients for the jump and the flux (this is because the coefficients are computed from boundary integrals involving $u$ on $\partial\Omega$). These problems and others were handled in the rest of our research. If you have any questions or desire further information, please feel free to contact Mathew Johnson at `majohnson3@bsu.edu`.

## References

[1] K. Bryan and M. Vogelius, *A review of selected works on crack identification*, Geometric Methods in Inverse Problems and PDE Control, IMA **137**, Springer (2004).

[2] J. Marsden and A. Tromba, Vector Calculus (4th Edition), Freeman (2000).