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Numerical Range of 3×3 Truncated Composition Operators

Maddison Guillaume and Amish Mishra

Maddison Guillaume graduated in 2019 with a Mathematics Education degree from Taylor University. She now works as a high school math teacher in Fishers, Indiana.

Amish Mishra holds a B.S. in Mathematics and Computer Science/Systems from Taylor University. As of 2019, he is a graduate student in the Mathematics Department of Florida Atlantic University.

Abstract In this paper, we look at the numerical ranges of complex truncated composition operators induced by analytic mappings of the open unit disk to itself. Our matrices of interest have dimension 3 and the corresponding operators are compact in the Hardy-Hilbert Space H^2. We investigate the geometric shapes of the numerical range for such operators.

1 Introduction

In an undergraduate Linear Algebra course, students generally view matrices as actions on vectors in $\mathbb{R}^n$. However, these vectors can also be viewed as coefficients of polynomials. For example, in $\mathbb{R}^3$, the vector $\begin{bmatrix} 2 & 3 & 5 \end{bmatrix}^T$ can be identified with the polynomial $2 + 3x + 5x^2$. In this way, $\mathbb{R}^3$ can be viewed as the space of polynomials degree 2 or less, denoted $\mathbb{P}_2$. In general, $\mathbb{R}^{n+1}$ can be viewed instead as $\mathbb{P}_n$. We keep the standard inner (dot) product the same, so we also now have a way to consider "angles" between and "lengths" of polynomials! In fact, taking this idea to infinite dimensions (and equating functions with their Taylor series - which are infinite polynomials) is the starting point for research on what is known as the Hardy-Hilbert space.

The Hardy-Hilbert space H^2 consists of functions analytic (continuously differentiable) in the open unit disk on the complex plane. What this means is that the functions have a Taylor series centered at the origin with a radius of convergence of at least 1. However, that is not the only requirement. For our analogy of dot product to make sense, we also require that any function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in H^2 must satisfy

$$\|f\|^2 = \langle f, f \rangle = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

This says that we can maintain our standard dot product, but the value becomes an infinite series that must converge. For example, $\frac{1}{2-z} = \frac{1}{2} + \frac{1}{4}z + \frac{1}{8}z^2 + \cdots$ becomes the vector $\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \ldots \end{bmatrix}^T$, and the inner product of that vector with itself is

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots + \frac{1}{4^n} + \cdots = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3} < \infty.$$

This natural generalization means that doing finite-dimensional work on $\mathbb{P}_n$ may give information about H^2, so open questions in H^2 are sometimes worth studying first on the simpler spaces $\mathbb{P}_n$. (In fact, these spaces are proper subspaces of H^2.) In particular, *composition operators* are an area of intense study on H^2. They have deep connections to multiplication operators (considered the building blocks of operator theory) and dynamical systems. **Note:** Although H^2 is defined over the complex numbers, the explanation above is the same: simply think of polynomials on $\mathbb{C}^n$ instead of $\mathbb{R}^n$.

A composition operator on H^2 with symbol φ is defined on any function f by $C_\varphi f = f \circ \varphi$. If φ maps the disk into itself and is analytic, then C_φ is a bounded (continuous) operator. For most composition operators, the subspaces $\mathbb{P}_n$ of H^2 are not invariant (meaning C_φ does not map $\mathbb{P}_n$ back into itself), but we can approximate the idea with a truncated matrix. We will call this a *truncated composition operator* (TCO). As an example, consider the polynomial space $\mathbb{P}_2$, with the standard basis now being represented as the functions $\{1, z, z^2\}$. If $\varphi = a + bz + cz^2$, then the composition matrix C_φ with respect to the standard basis is

$$\begin{bmatrix} 1 & a & a^2 \\ 0 & b & 2ab \\ 0 & c & b^2 + 2ac \end{bmatrix}.$$

This is achieved by considering the action of the composition operator, functionally, on each of the basis elements, and then writing out the matrix. So, $C_\varphi 1 = 1 \circ \varphi = 1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, and this is true for any composition operator. Likewise, $C_\varphi z = z \circ \varphi = \varphi = a + bz + cz^2 = \begin{bmatrix} a & b & c \end{bmatrix}^T$, and $C_\varphi z^2 = z^2 \circ \varphi = \varphi^2 = a^2 + 2abz + (b^2 + 2ac)z^2 + \cdots$, but here, we *truncate* the terms past degree 2, giving the vector $\begin{bmatrix} a^2 & 2ab & b^2 + 2ac \end{bmatrix}^T$. Since we truncated at degree 2 for $\mathbb{P}_2$, we will call this operator $P_2 C_\varphi$.

Now that we have described the setting, what would we like to know about composition operators on H^2? The numerical range of an operator T, $W(T)$, is defined by

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}.$$

The Toeplitz-Hausdorff Theorem states that this is a convex subset of the complex plane that gives information about the operator. For example, the closure of the numerical range always contains the convex hull of the eigenvalues, and so can be used to approximate or find eigenvalues. In infinite-dimensional settings, the numerical range can indicate when an operator commutes with, or is equal to, its adjoint. Recently, numerical range has been applied to quantum physics and quantum computing. However, in the case of composition operators on H^2, incredibly little is known about the numerical range–even composition operators with linear symbols have not had their numerical ranges discovered! Therefore, we will focus on the numerical ranges of our truncations of these composition operators on $\mathbb{P}_2$.

In the second section of this paper, we will investigate the numerical range of every possible 3×3 truncated composition operator (TCO) and give proofs for the shapes for which we did not need projective geometry. In section three, we will give equations for the 3×3 TCOs for which we use projective geometry to identify their curves. In the final section, we will pose questions for further research. Throughout this paper, we display images for the numerical range of an operator; these pictures are thanks to Valentin Matache's MATLAB function that plots the numerical range [6]. The other plots in this paper were constructed using our own MATLAB, Mathematica, and Desmos code.

2 3×3 Numerical Range

For $P_n C_\varphi$ when $n = 2$, there are seven non-trivial possibilities for the form of φ. This is because $P_2 C_\varphi$ truncates all terms past z^2; the infinite series for $\varphi = a + bz + cz^2 + dz^3 + \cdots$ is truncated to only its first three terms $a + bz + cz^2$, where a, b and c are constants in $\mathbb{C}$. We have listed each possible such φ and the shape of the numerical range of the corresponding TCO in Table 1 below. We now set out to prove each case. We begin by proving the numerical range of $\varphi = bz$ is a line or triangle. This relies on the well-known fact that the numerical range of a normal matrix is the convex hull of the eigenvalues [1].

Theorem 1. *Let T be the matrix representation of $P_2 C_\varphi$ with $\varphi = bz$. When $b \in \mathbb{R}$, $W(T)$ is a degenerate triangle (a line segment) with endpoints at 1 and b^2. When $b \notin \mathbb{R}$, $W(T)$ is a triangle with vertices at 1, b, and b^2.*

Proof. When $\varphi = bz$ for C_φ and $b \neq 0$, our truncated matrix is

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b^2 \end{bmatrix}.$$

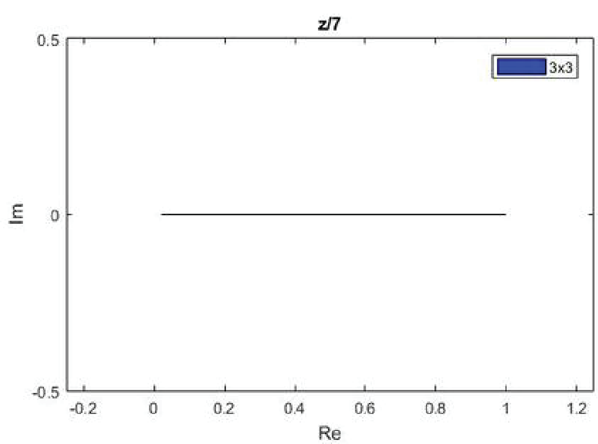
φ	Numerical Range Description
$\varphi = bz$	Triangle or Line
$\varphi = bz + cz^2$	Convex Hull of an Ellipse and a Point
$\varphi = cz^2$	Convex Hull of an Ellipse and a Point
$\varphi = a$	Ellipse
$\varphi = a + bz + cz^2$	Convex Hull of the Linear Envelope of $p_t(x, y) = 0$
$\varphi = a + cz^2$	Convex Hull of the Linear Envelope of $p_t(x, y) = 0$
$\varphi = a + bz$	Convex Hull of the Linear Envelope of $p_t(x, y) = 0$

Table 1: Numerical Range Description of 3×3 TCOs where $a, b, c \neq 0$

Since T is a diagonal matrix, $T^*T = TT^*$. Therefore, T is normal. Since T is normal, $W(T)$ is the convex hull of the eigenvalues of T. The eigenvalues of T are $1, b$ and b^2.

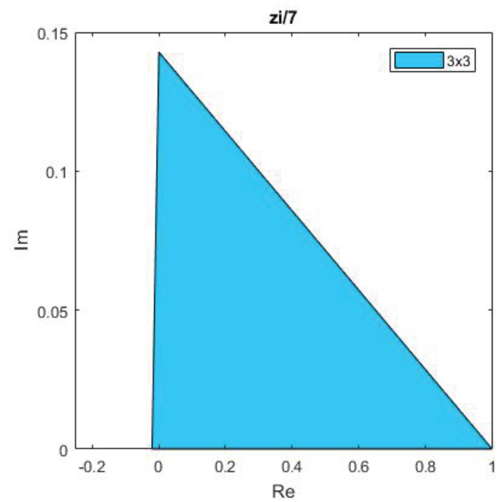
When $b \notin \mathbb{R}$, the eigenvalues are no longer collinear, and the convex hull of the eigenvalues of T will form a triangle with the vertices at the eigenvalues $1, b$, and b^2. When $b \in \mathbb{R}$, the eigenvalues of T are collinear, and the convex hull of the eigenvalues will be a line. Therefore, with $b \in \mathbb{R}$, $W(T)$ degenerates into a line segment with endpoints at 1 and b^2, the maximum and minimum eigenvalues respectively. □

Figure 1: Numerical Range of P_2C_φ when $\varphi = z/7$



As seen from Theorem 1, Figure 1 is an image of the numerical range of P_2C_φ when $\varphi = bz$ such that $b = \frac{1}{7}$. Note that the numerical range is a line segment with endpoints at 1 and $\frac{1}{49}$. Figure 2 shows the numerical range of C_φ when $\varphi = bz$ such that $b = \frac{i}{7}$. Note that the numerical range is a triangle with vertices at $1, \frac{i}{7}$ and $\frac{-1}{49}$.

Figure 2: Numerical Range of P_2C_φ when $\varphi = iz/7$



From previous research, we can also classify when the numerical range is the convex hull of an ellipse and a point. We use the following theorem to do so.

Theorem 2 (Theorem 2.3 of [5]). *The numerical range $W(T)$ of a 3×3 matrix T with the eigenvalues λ_j, $j = 1, 2, 3$, is the convex hull of an ellipse and a point if and only if*

(i) T is not normal, that is, $d := \mathrm{Tr}(T^*T) - \sum_{j=1}^{3}|\lambda_j|^2 \neq 0$ and

(ii) the number

$$\lambda := \mathrm{Tr}(T) + \tfrac{1}{d}\left(\sum_{j=1}^{3}|\lambda_j|^2\lambda_j - \mathrm{Tr}(T^*T^2)\right)$$

coincides with one of λ_j.

If these conditions hold, then in $W(T)$, the point of the convex hull is at $p = \lambda$, while the minor axis of the ellipse has length $\sqrt{d}$ and its foci coincide with the two remaining eigenvalues, λ_1 and λ_2, of T. In particular, $W(T)$ is a true ellipse if and only if

$$(|\lambda_1 - \lambda| + |\lambda_2 - \lambda|)^2 - |\lambda_1 - \lambda_2|^2 \leq d.$$

The following corollaries follow directly from this theorem.

Corollary 3. *Let T be the matrix representation of P_2C_φ with $\varphi = bz + cz^2$ such that $c \neq 0$. Then $W(T)$ is the convex hull of an ellipse and a point.*

Proof. Suppose $\varphi = bz + cz^2$ for C_φ such that $c \neq 0$. Our truncated matrix is

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & c & b^2 \end{bmatrix}.$$

Computing λ and d, we see

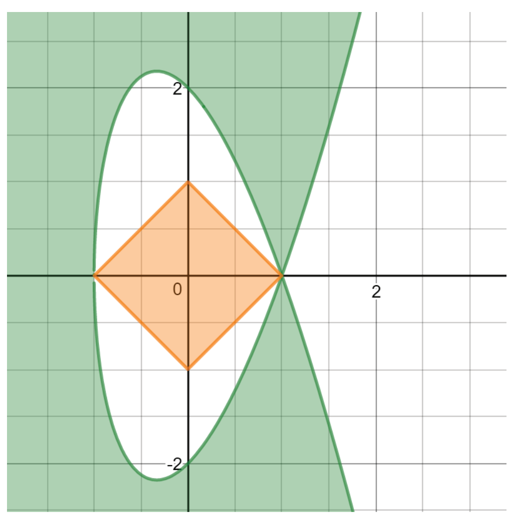
$$\lambda = 1 + b + b^2 + \frac{b|b|^2 + b^2|b|^4 - b^4\overline{b}^2 - b^2c\overline{c} - b^2\overline{b} + bc\overline{c}}{-|b|^2 - |b|^4 + b\overline{b} + b^2\overline{b}^2 + c\overline{c}}$$

and

$$d = -|b|^2 - |b|^4 + b\overline{b} + b^2\overline{b}^2 + c\overline{c}.$$

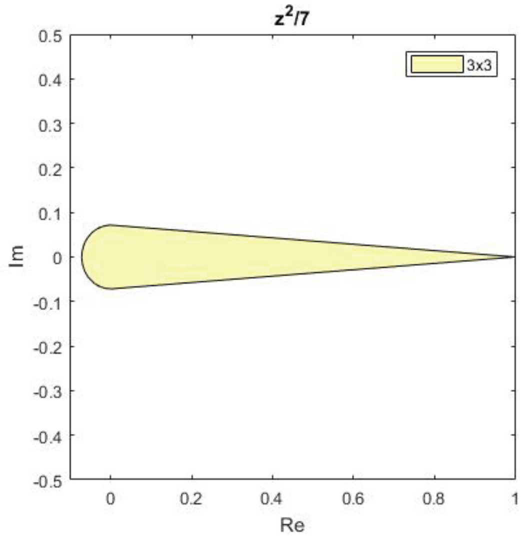
Using the fact that $x\overline{x} = |x|^2$ for $x \in \mathbb{C}$, we see $\lambda = 1$, and $d = |c|^2$. The eigenvalues of our matrix are $\lambda_1 = 1$, $\lambda_2 = b$ and $\lambda_3 = b^2$. Since $\lambda = \lambda_1$, $W(T)$ is the convex hull of an ellipse and a point by Theorem 2. Also by Theorem 2, $W(T)$ is a true ellipse if $(|b-1| + |b^2-1|)^2 - |b^2-b|^2 \leq |c|^2$. But, since $|b| + |c| < 1$ in order for the corresponding operator on H^2 to be compact, $(|b-1| + |b^2-1|)^2 - |b^2-b|^2$ will never be less than $|c|^2$ (see Figure 3). Therefore, $W(T)$ is always the convex hull of an ellipse and a point. Note that the case $b = 0$ is a sub-case for this corollary: the numerical range of the corresponding TCO is the convex hull of a circle centered at 0 and a point at 1. □

Figure 3: When $(|b-1| + |b^2-1|)^2 - |b^2-b|^2 \leq |c|^2$ and $|b| + |c| < 1$



In Figure 4, we show the numerical range of P_2C_φ when $\varphi = cz^2$ such that $c = \frac{1}{7}$. It is the convex hull of a circle whose center is at 0 and a point at 1.

Figure 4: Convex Hull of an Ellipse and a Point: Numerical Range of P_2C_φ when $\varphi = z^2/7$



We conclude this section with a final corollary on the numerical range of P_2C_φ with $\varphi = a$.

Corollary 4. *Let T be the matrix representation of P_2C_φ with $\varphi = a$ such that $a \neq 0$. $W(T)$ is an ellipse with foci at 0 and 1.*

Proof. When $\varphi = a$ for C_φ and $a \neq 0$, our truncated matrix is

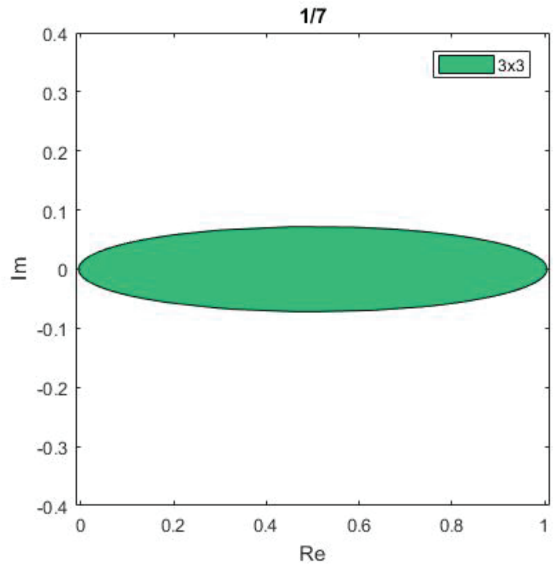
$$T = \begin{bmatrix} 1 & a & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Computing λ and d, we see $\lambda = 1 + \frac{-a\overline{a} - a^2\overline{a}^2}{a\overline{a} + a^2\overline{a}^2}$ and $d = a\overline{a} + a^2\overline{a}^2$. Using the fact that $x\overline{x} = |x|^2$ for $x \in \mathbb{C}$, we see $\lambda = 0$ and $d = |a|^2(1 + |a|^2)$. The eigenvalues of T are 1 and 0. Since $\lambda = 0$, λ coincides with an eigenvalue of T. Therefore, $W(T)$ is the convex hull of an ellipse and a point by Theorem 2. We now check the criteria for a true ellipse.

In our case, $\lambda = 0$. So, $(|\lambda_1 - \lambda| + |\lambda_2 - \lambda|)^2 - |\lambda_1 - \lambda_2|^2 = (|1| + |0|)^2 - |1 - 0|^2 = 0$. Also, $d = |a|^2(1 + |a|^2) > 0$. Since $0 < |a|^2(1 + |a|^2)$, we know that $W(T)$ is a true ellipse with foci at the two eigenvalues by Theorem 2. Therefore, $W(T)$ is a true ellipse with foci at 1 and 0. □

Figure 5 shows the numerical range of P_2C_φ when $\varphi = a$ such that $a = 1/7$. The foci of the ellipse are at 0 and 1.

Figure 5: Elliptical Numerical Range of $P_2 C_\varphi$ when $\varphi = 1/7$



3 3×3 Cases with Projective Geometry

For the remaining three cases, we reference the work of Kippenhahn [2], who made groundbreaking discoveries about numerical range in the late 1950s. We are only able to describe the boundary curves of these numerical ranges using projective geometry. We focus on some specific examples to show explicitly how some of these shapes look. In a paper by Mihaela and Valentin Matache referencing Kippenhahn's work [4], the following theorem is restated.

Theorem 5 (Theorem 1 of [4])**.** *Let T be any square matrix with complex entries. Its numerical range $W(T)$ is the convex hull of the linear envelope of the curve having the equation $p_t(x,y) = 0$, where*

$$p_t(x,y) = det(x\mathscr{R}T + y\mathscr{I}T + I).$$

The linear envelope is the family of lines that are tangent to a given curve. In Matache's definition, we find where $p_t(x,y) = 0$, which is known as the dual to the numerical range. Then, we take every pair (X,Y) on the dual curve and graph the lines $Xx + Yy + 1 = 0$. If we take the convex hull of the linear envelope, we obtain the numerical range. When $\varphi = a + bz + cz^2$ such that $a,b,c \neq 0$, $a,c \neq 0$, or $a,b \neq 0$, for C_φ, we get the following dual equation:

$$p_t(x,y) = \tfrac{1}{4}a^4bx^3 + \tfrac{1}{4}a^4bxy^2 - \tfrac{a^4x^2}{4} - \tfrac{a^4y^2}{4} - \tfrac{1}{4}a^3cx^3 - \tfrac{1}{4}a^3cxy^2 - \tfrac{5}{4}a^2b^2x^3 - a^2b^2x^2 - \tfrac{5}{4}a^2b^2xy^2 - a^2b^2y^2 - \tfrac{a^2x^2}{4} - \tfrac{a^2y^2}{4} + abcx^3 + abcx^2 + abcxy^2 + abcy^2 + 2acx^2 + 2acx + b^3x^3 + b^3x^2 + b^2x^2 + b^2x + bx^2 + bx - \tfrac{c^2x^3}{4} - \tfrac{c^2x^2}{4} - \tfrac{1}{4}c^2xy^2 - \tfrac{c^2y^2}{4} + x + 1.$$

We will now give an example of the numerical range as a convex hull of the linear envelope of the curve $p_t(x,y) = 0$. We use the equation $\varphi = 1/7 + z/7$. For φ, we begin by finding the equation

$$p_t(x,y) = \frac{81x^3 + 5397x^2 - 17xy^2 + 39102x - 189y^2 + 33614}{33614}$$

using the formula above. Setting $p_t(x,y) = 0$, we get the graph of the dual found in Figure 6. The graph of the lines $Xx + Yy + 1 = 0$ with (X,Y) on the dual curve is found in Figure 7. In Figure 8, we see the convex hull of the linear envelope, which is the final numerical range picture.

Figure 6: $p_t(x,y) = 0$ when C_φ when $\varphi = 1/7 + z/7$

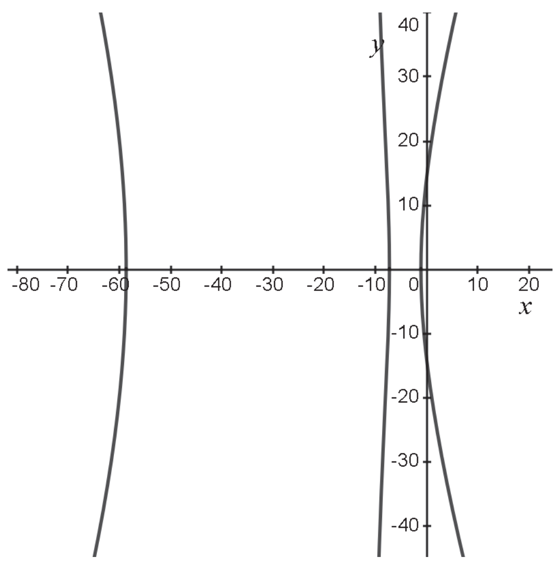


Figure 7: $Xx + Yy + 1 = 0$ from $p_t(x,y) = 0$ of C_φ when $\varphi = 1/7 + z/7$

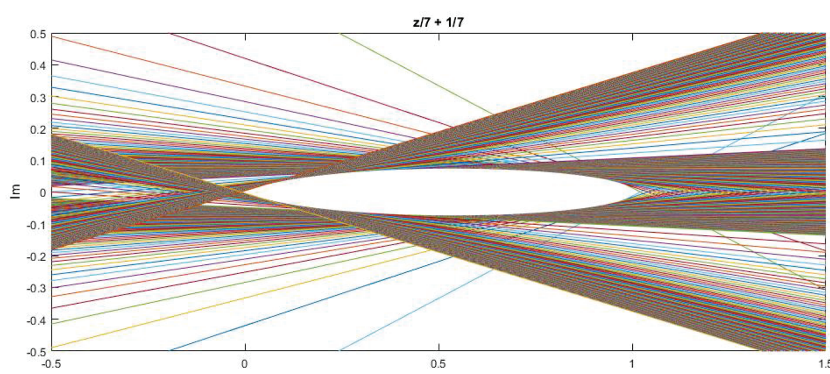
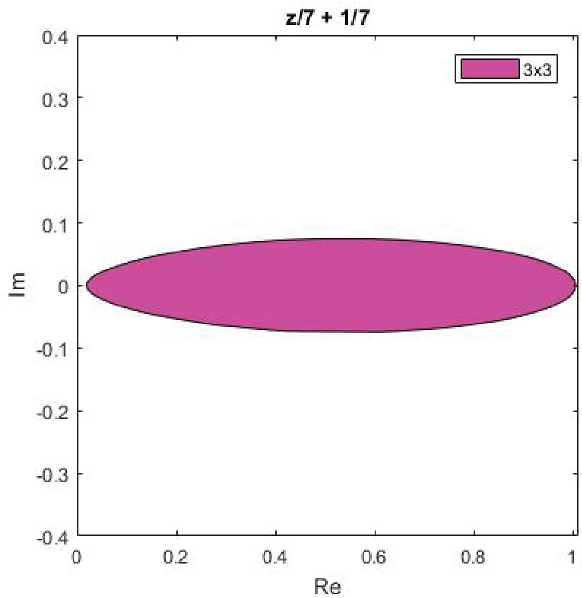


Figure 8: Numerical Range of $P_2 C_\varphi$ when $\varphi = 1/7 + z/7$



Close visual inspection of Figure 7 demonstrates that the numerical range is not completely elliptical. In fact, there is only one case where the numerical range of $P_2 C_\varphi$ when $\varphi = a + az$ is an ellipse. Computing λ and d from Theorem 2 yields $\lambda = -\frac{(a-5)a^2}{5a^2+1}$ and $d = 5a^4 + a^2$. The only case where λ coincides with one of the eigenvalues is when $a = 1/3$. When this is the case, $\varphi = 1/3 + z/3$ and our TCO is

$$T = \begin{bmatrix} 1 & 1/3 & 1/9 \\ 0 & 1/3 & 2/9 \\ 0 & 0 & 1/9 \end{bmatrix}.$$

We compute and find $\lambda = 1/3$, and $d = 14/81$. So, we have shown that the numerical range of $P_2 C_\varphi$ when $\varphi = 1/3 + z/3$ is the convex hull between an ellipse and a point. Furthermore, to check if $W(T)$ is an ellipse, we compute $(|\lambda_1 - \lambda| + |\lambda_2 - \lambda|)^2 - |\lambda_1 - \lambda_2|^2 \leq d$. We compute each side of the inequality and see that it is satisfied as $0 \leq 14/81$. Therefore, $W(T)$ is a true ellipse with foci at 1 and 1/9. See Figure 10 and Figure 11 for numerical range pictures of $P_2 C_\varphi$ when $\varphi = 1/3 + z/3$.

Figure 9: $p_t(x,y) = 0$ when C_φ when $\varphi = 1/3 + z/3$

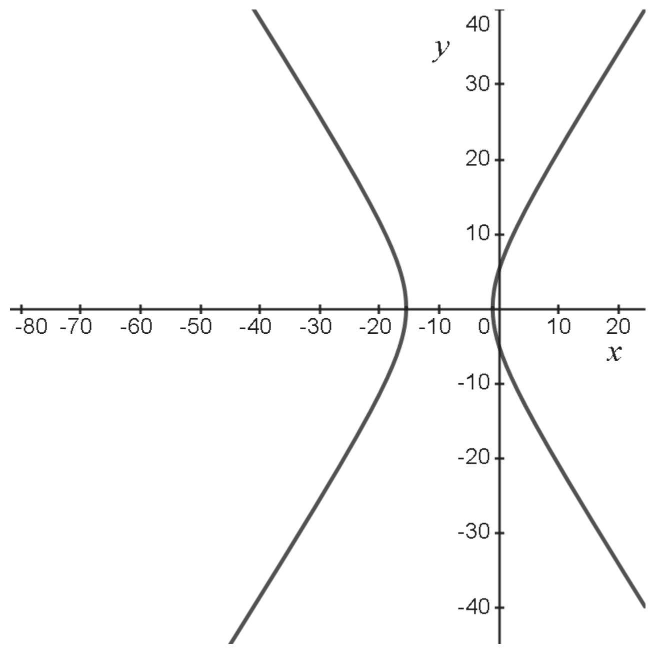


Figure 10: The Linear Envelope of $p_t(x,y) = 0$ of C_φ when $\varphi = 1/3 + z/3$

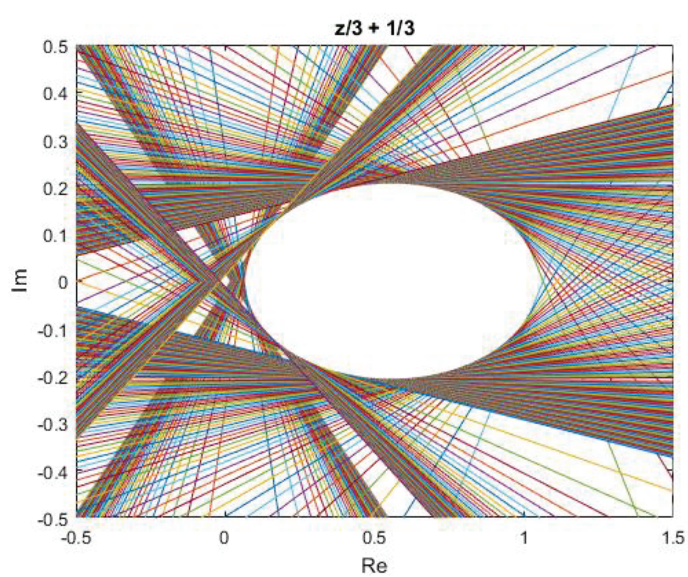
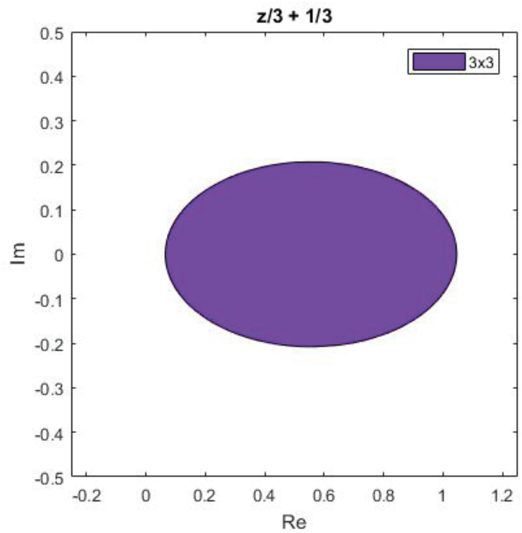


Figure 11: Numerical range of $P_2 C_\varphi$ when $\varphi = 1/3 + z/3$



4 Final Remarks and Further Questions

We conclude our paper with further questions to investigate.

1. Are there any classifications of numerical ranges for the 3×3 cases that extend to infinite dimensions? If so, is there a way to determine the limit of the numerical radius?

2. To what extent can our methods be used on 4×4 matrices or higher? When does the numerical range cease to look like one of our described shapes as the Taylor series differ more for larger truncations (if it does)?

Bibliography

[1] Gallier, Jean. (2011). *Geometric Methods and Applications For Computer Science and Engineering*. New York, NY. Springer Sceince+Buisness Media.

[2] Kippenhahn, Ruldolf (Translated by Paul F. Zachlin and Michiel E. Hochstenbach), *On the numerical range of a matrix*, Linear and Multilinear Algebra, **56**(1-2) (2008), 185-225.

[3] Martińez-Avendanõ, R.A., Rosenthal, P. (2007). *An Introduction to Operators on the Hardy-Hilbert Space*. New York, NY. Springer Sceince+Business Media.

[4] Matache, M. T. Matache, V., *When is the numerical range of a nilpotent matrix circular?*, Appl. Math. and Computation, **216**(1) (2010), 269-275.

[5] Rault, P. X., Sendova, T., Spitkovsky, I. M., *3-By-3 Matrices with Elliptical Numerical Range Revisited*, Electron. J. Linear Algebra **26** (2013), 158-167.

[6] Valentin Matache (29 Nov, 2001) nrange (Version 1.0) [MATLAB function]. https://www.mathworks.com/matlabcentral/fileexchange/1085-nrange.