Ball State Undergraduate Mathematics Exchange
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Vol. 11, No. 1 (Fall 2017)
Pages 28 – 39

Properties of permutations containing exactly one 123 subsequence

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Abstract We explore new details about permutations with exactly one increasing subsequence of length three in increasing generality. We first examine some special

cases of these permutations when we specify the positions of the increasing subsequence and give a conjecture on one of these cases. Then we present a bijection between permutations with one increasing subsequence of length three and triples of Dyck paths, which as a corollary gives an enumeration result previously proved only with generating functions.

Introduction

A permutation σ is given by $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ where $|\sigma| = n$. We define a 123 pattern as any increasing subsequence of σ of length three, i.e., a 123 pattern exists in a permutation σ if there exists $a < b < c$ such that $\sigma_a < \sigma_b < \sigma_c$. For example, $\sigma = 62845731$ has four occurrences of a 123 pattern: 245, 247, 257, and 457. We will be concerned with permutations containing exactly one occurrence of a 123 pattern. In this case we will let $\sigma_a = i$, $\sigma_b = j$, $\sigma_c = k$ for some $i < j < k$, and in the rest of this paper, i, j, k, a, b, c are reserved for this meaning.

Definition 1. The set $\Phi_n^{i,j,k} = \{\sigma \in S_n \mid \sigma$ contains exactly one increasing subsequence of length three and that subsequence is $ijk\}$. For example, $\sigma = 972856413$ is a member of $\Phi_9^{2,5,6}$. We also let $\Phi_n = \bigcup_{i,j,k} \Phi_n^{i,j,k}$

The set Φ_n has been studied by others preceding our investigation. For small n, $|\Phi_n|$ produces the sequence $1, 6, 27, 110, 429, 1638, \ldots$; this is sequence A003517 in the On-line Encyclopedia of Integer Sequences [8]. In [5], Noonan proves that the number of these σ in Φ_n is $\frac{3}{n}\binom{2n}{n+3}$.

In the next section, we describe the ways to count the permutations in $\Phi_n^{i,j,k}$. We investigate those σ in $\Phi_n^{j-1,j,j+1}$, $\Phi_n^{1,j,j+1}$, and $\Phi_n^{j-1,j,k}$. In the section Bijection between Φ_n and triples of Dyck paths, we present a bijection ϕ between those σ in Φ_n and triples of Dyck paths. Permutations in Φ_n are represented first with two Dyck paths, one which starts with two up steps and another which ends in two down steps. These two are then further broken into a total of six Dyck paths and finally recombined into three Dyck paths. As a corollary, this bijection shows Φ_n is enumerated by the sum of the product of three Catalan numbers $C_r C_s C_t$ where $r, s, t \geq 1$ and $r + t + s = n$. Previously this was proved using generating functions, but ours is a bijective proof.

Counting the permutations with exactly one increasing subsequence of length 3

Definition 2. For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, the *reverse of* σ, denoted σ^r, is defined as $\sigma^r = \sigma_n\sigma_{n-1}\cdots\sigma_1$. The *complement of* σ, denoted σ^c, is defined as $\sigma^c = (n+1-\sigma_1)(n+1-\sigma_2)\cdots(n+1-\sigma_n)$.

Note the subsequence $(n+1-i)(n+1-j)(n+1-k)$ is the complement of the subsequence ijk for σ of length n.

Theorem 3. *The number of σ in $\Phi_n^{i,j,k}$ equals the number of σ in $\Phi_n^{n+1-i,n+1-j,n+1-k}$.*

Proof. If we apply the reverse map to σ, we find σ^r has exactly one occurrence of 321, and that occurrence is kji. In σ^r, $k = \sigma_{n+1-c}$, $j = \sigma_{n+1-b}$, and $i = \sigma_{n+1-a}$. If we apply the complement map to σ^r, now $\sigma_{n+1-c} = n+1-k$, $\sigma_{n+1-b} = n+1-j$, and $\sigma_{n+1-a} = n+1-i$ by definition of the complement. Since $n+1-k < n+1-j < n+1-i$ and

$n+1-c < n+1-b < n+1-a$, σ^{rc} has exactly one occurrence of a 123 pattern, namely $\sigma_{n+1-c}\sigma_{n+1-b}\sigma_{n+1-a}$. This shows there is a one-to-one correspondence between each σ in $\Phi_n^{i,j,k}$ and each σ in $\Phi_n^{n+1-i,n+1-j,n+1-k}$. Therefore, the number of σ in $\Phi_n^{i,j,k}$ is equal to the number of σ in $\Phi_n^{n+1-i,n+1-j,n+1-k}$. □

Lemma 4. *For permutations in $\Phi_n^{i,j,k}$, j is always in a fixed position, and that position is $n-j+1$, i.e., $j = \sigma_{n-j+1}$. All elements q such that $q < j$, except i, and k appear after j. All elements m such that $j < m \leq n$, except k, and i appear before j.*

Proof. Let σ be a permutation in $\Phi_n^{i,j,k}$. In order to avoid another occurrence of 123, all elements q such that $q < j$, excluding i, must appear after j in σ. There are $j-2$ of these elements, and together with k, there are at least $j-2+1 = j-1$ elements following j. In order to avoid another occurrence of 123, all elements m such that $j < m \leq n$ except k must appear before j in σ. There are $n-j-1$ of these elements plus i. Then there are at least $n-j-1+1 = n-j$ elements appearing before j in σ. Since there are at least $j-1$ elements after j and at least $n-j$ elements before j, then $j = \sigma_{n-j+1}$. □

Definition 5. Let $\sigma \in \phi_n^{i,j,k}$. We define the *head* of permutation σ as $\sigma_1 \cdots \sigma_{n-j}$, i.e., there are $n-j$ elements in the head. The *tail* of the permutation we define as $\sigma_{n-j+2} \cdots \sigma_n$, i.e., there are $j-1$ elements in the tail.

For example, $\sigma = 972856413$ has a head of 9728 and a tail of 6413. By Lemma 4, $j = \sigma_{n-j+1}$, so j is between the head and the tail and is contained in neither. The head, j, and the tail completely describe each σ.

Increasing subsequence of the form $(j-1)(j)(j+1)$

In this section we restrict our investigation to those σ in $\Phi_n^{j-1,j,j+1}$. For example, $\sigma = 76384215$ is a member of $\Phi_n^{j-1,j,j+1}$ where $j = 4$ because it has an increasing subsequence of length three that is 345. By Theorem 3, this is the same as investigating those σ in $\Phi_n^{n-j,n+1-j,n+2-j}$.

Lemma 6. *For all σ in $\Phi_n^{j-1,j,j+1}$, there are C_{n-j} ways to arrange the head of σ.*

Proof. By Definition 5, there are $n-j$ elements in the head of the permutation. All elements in the head must be greater than $j+1$ except for $j-1$ by Lemma 4. Elements in the head can be arranged in any way avoiding another 123 pattern because none of the numbers in the head could create another 123 pattern with j or any of the elements in the tail. It is known the Catalan numbers, C_n, count the ways to arrange permutations of length n avoiding 123 patterns [4]. Therefore, there are C_{n-j} ways to arrange the head of the permutation σ. □

Lemma 7. *For all σ in $\Phi_n^{j-1,j,j+1}$, there are C_{j-1} ways to arrange the tail of σ.*

Proof. By Definition 5, there are $j-1$ elements in the tail of the permutation. All elements in the tail must be less than $j-1$ except for $j+1$ by Lemma 4. Elements in the tail can be arranged in any way that avoids another 123 pattern since none of the numbers in the tail could create another 123 pattern with j or with any of the elements in the head. Once again, there are C_n ways to arrange permutations of length n that avoid a 123 pattern, thus there are C_{j-1} ways to arrange the tail of the permutation σ. □

Theorem 8. *For all $n \geq 3$ and all j with $1 < j < n$, the number of permutations in $\Phi_n^{j-1,j,j+1}$ is $C_{n-j}C_{j-1}$.*

Proof. Lemmas 6 and 7 completely describe those σ in $\Phi_n^{j-1,j,j+1}$. Therefore the number of σ in $\Phi_n^{j-1,j,j+1}$ is $C_{n-j}C_{j-1}$. $\square$

For example, $|\Phi_8^{3,4,5}| = C_{8-4}C_{4-1} = C_4C_3 = 14 \times 5 = 70$, i.e., there are 70 distinct permutations in $\Phi_8^{3,4,5}$.

Increasing subsequence of the form $1(j)(j+1)$

In this section we restrict our investigation to those σ in $\Phi_n^{1,j,j+1}$. For example, σ = 71864532 is a member of $\Phi_n^{1,j,j+1}$ where $j = 4$ because it has an increasing subsequence of length three that is 145. By Theorem 3, this is the same as investigating those σ in $\Phi_n^{n-j,n-j+1,n}$.

Lemma 9. *Every σ in $\Phi_n^{1,j,j+1}$ will always have a fixed tail where $\sigma_{n-j+2} = j+1$, $\sigma_{n-j+3} = j-1$, $\sigma_{n-j+4} = j-2$, …, $\sigma_n = 2$ for all $j > 2$. If $j = 2$, then $\sigma_n = 3$.*

Proof. Consider σ in $\Phi_n^{1,j,j+1}$. All elements f such that $1 < f < j$ and $j+1$ must appear in the tail of the permutation by Lemma 4. By way of contradiction let $\sigma_a = n$ and $\sigma_b = m$ for $a < b$ where m, n are elements in the tail such that $n < m$. If so, an additional occurrence of a 123 pattern is formed by $1(n)(m)$. This contradicts our original assumption of only one occurrence of a 123 pattern. Therefore, all elements in the tail appear in decreasing order. Hence, every σ in $\Phi_n^{1,j,j+1}$ has a fixed tail where $\sigma_{n-j+2} = j+1$, $\sigma_{n-j+3} = j-1$, $\sigma_{n-j+4} = j-2$, …, $\sigma_n = 2$ for all $j > 2$. If $j = 2$, the tail only contains the element 3, so $\sigma_n = 3$. $\square$

Theorem 10. *For all $n \geq 3$ and all j such that $1 < j < n$, the number of permutations in $\Phi_n^{1,j,j+1}$ is C_{n-j}.*

Proof. By Lemma 4 we know elements of σ in $\Phi_n^{1,j,j+1}$ that are greater than $j+1$, and 1, must appear in the head. Then there are $n-(j+1)+1 = n-j$ elements in the head. By definition of the head, these are all of the elements in the head. Since we know the tail is fixed by Lemma 9, the only elements that can be rearranged are the elements in the head. We know there are C_{n-j} ways to arrange these elements to avoid a 123 pattern since the elements in the head cannot form another 123 pattern with the elements in the tail. Therefore, the number of σ in $\Phi_n^{1,j,j+1}$ is C_{n-j}. $\square$

For example, $|\Phi_9^{1,5,6}| = C_{9-5} = C_4 = 14$, i.e., there are 14 distinct σ in $\Phi_9^{1,5,6}$.

Increasing subsequence of the form $(j-1)jk$

In this section we restrict our investigation to those σ in $\Phi_n^{j-1,j,k}$. For example, σ = 865394712 is a member of $\Phi_n^{j-1,j,k}$ where $j = 4$ because it has an increasing subsequence of length 3 that is 347. By Theorem 3 this is the same as investigating those σ in $\Phi_n^{n-k+1,n-j+1,n-j+2}$.

By Lemma 4 all elements less than $j-1$ along with k are in the tail of the permutation. So there are $j-1$ elements in the tail. By definition of the tail, these are all of the elements in the tail. Therefore, there are C_{j-1} ways to arrange the elements in the tail since none of these elements can form another 123 pattern with the elements in the head.

All elements m where $m > k$, all elements q where $j < q < k$, and $(j-1)$ must go in the head of the permutation by Lemma 4, i.e., all elements greater than j, excluding k and including $j-1$ are in the head. In order to avoid a 123 pattern in the head of the permutation, we know all elements q as well as $(j-1)$ must be fixed in decreasing order. To find the formula counting the ways to arrange the head, we can treat all elements q and $(j-1)$ as 1s since they are all in a fixed order and are less than all elements m. Now we look for a way to arrange words of length n which have r 1s that avoid a 123 pattern where n is the length of the head.

Conjecture 11. *The number of words of length n with r 1s that avoid the pattern 123 is given by the recurrence relation:*

$$a_{r,n} = a_{r-1,n-1} + a_{r+1,n}.$$

$$
\begin{array}{lccccccccccccc}
n=1 & & & & & & & 1 & & & & & & \\
n=2 & & & & & & 1 & & 2 & & & & & \\
n=3 & & & & & 1 & & 3 & & 5 & & & & \\
n=4 & & & & 1 & & 4 & & 9 & & 14 & & & \\
n=5 & & & 1 & & 5 & & 14 & & 28 & & 42 & & \\
n=6 & & 1 & & 6 & & 20 & & 48 & & 90 & & 132 & \\
n=7 & 1 & & 7 & & 27 & & 75 & & 165 & & 297 & & 429
\end{array}
$$

Figure 1: Triangle for the recurrence relation $a_{r,n} = a_{r-1,n-1} + a_{r+1,n}$.

We conjecture that the triangle in Figure 1 above represents the number of words containing r 1s and have length n that avoid 123. We found some of the terms in the triangle through examples and then found the completed triangle in a paper written by Noonan and Zeilberger [6]. Each row corresponds to the length of the word. Each diagonal that starts on the top left and goes to the bottom right corresponds to the number of 1s in the word where the diagonal 1 2 5 14 42 132 429 corresponds to $r = 1$. The recurrence relation in Conjecture 11 was not found in Noonan and Zeilberger's paper.

In Conjecture 11, $a_{r-1,n-1}$ represents the words in the set of $a_{r,n}$ words where the two leftmost 1s are in adjacent positions. Because the two 1s are next to each other, they can be treated as one 1. Thus, the length of the word would decrease to $n-1$ and the number of 1s would decrease to $r-1$. $a_{r+1,n}$ represents the remaining words in the set of $a_{r,n}$ words in which the two left most 1s are not adjacent. Note that this is only one approach to solving Conjecture 11. It is possible that there are others.

Bijection between Φ_n and triples of Dyck paths

Definition 12. Let D_n be the set of all Dyck paths with n up steps. Also, let $TD_n = \{D_r \times D_s \times D_t \mid r, s, t \geq 1, r+s+t = n\}$.

The main goal of this section will be to present a bijection between Φ_n and TD_n. We begin with some background information.

It is known that a Dyck path, P, has a first return decomposition of the form $P = U\alpha D\beta$ where α and β are (possibly empty) Dyck paths [1]. This decomposition is unique. Figure 2 illustrates this decomposition.

A permutation can be easily represented as an $n \times n$ array. We place $\sigma_g = h$ in the array at (g, h). To construct a Dyck path from a 123−avoiding permutation, we

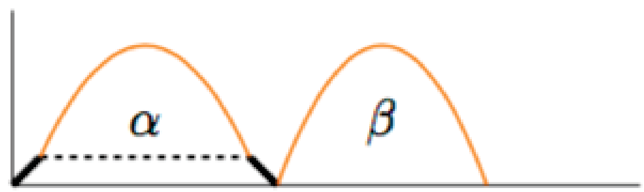

Figure 2: The first return decomposition $U\alpha D\beta$ of a Dyck path P [1].

follow the bijection introduced by Elizalde and Deutsch [3]. We begin at $(0, n)$ and bound our permutation with a sequence of down and right steps. Figure 3 demonstrates $\sigma = 4231$ and its corresponding Dyck path.

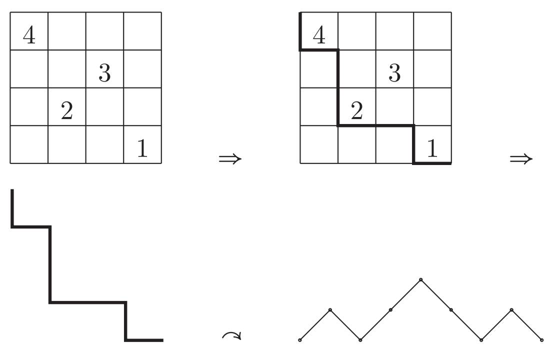

Figure 3: The permutation $\sigma = 4231$ and its Dyck path $UDUUDDUD$.

Lemma 13. *For all σ in $\Phi_n^{i,j,k}$, when σ is represented as an $n \times n$ array, j lies on the line that begins at point $(0, n)$ and ends at $(n, 0)$, which we will call the diagonal.*

Proof. In order for element d of permutation σ to lie on the diagonal, d must be in the position $n - d + 1$ of the permutation, i.e. $d = \sigma_{n-d+1}$. By Lemma 4, we know this is true for all j in σ in $\Phi_n^{i,j,k}$. $\square$

Theorem 14. *There is a bijection $\phi : \Phi_n \to TD_n$.*

Proof. We begin by creating two Dyck paths from σ in $\Phi_n^{i,j,k}$. We first plot σ on an $n \times n$ array, as previously described. We then place two rectangles in the array: one at (σ_a, j) and the other at (σ_b, i). Then, starting from the top leftmost box in the array $(0, n)$, we draw a path to the bottom rightmost box in the array $(n, 0)$ by only using down steps and right steps. All of the numbers and rectangles in the array except for i that are below the diagonal must be to the right of the path, while the path remains as close to the diagonal as possible. Once the path has passed the first rectangle, we finish the first Dyck path by extending the path to the diagonal. Since the path extends underneath the rectangle, and then has to extend by at least one more square to the diagonal, this first path is guaranteed to end in two right steps.

We start the second Dyck path at the diagonal immediately to the left of j and build the path the same way as previously discussed. The path extends down from the diagonal at least one step to get to the top of the rectangle and then one more down step to get to the bottom of the rectangle. Therefore, the second path begins in at least two down steps. We end the second Dyck path once we reach $(n, 0)$. Next we take away the grid lines, numbers, and rectangles, separate the two distinct Dyck paths, and rotate the paths 135° clockwise. After doing so, we see the right steps translate to up steps. So it is guaranteed the Dyck path on the left will always end with two down steps, and the Dyck path on the right will always start with two up steps. Note that essentially we have modified our permutation with one occurrence of a 123 pattern and created a permutation with no 123 patterns by switching the positions of the 2 and 1 in the pattern, and then applying the bijection of Elizalde and Deutsch [3], creating two paths instead of one, which marks the position where the 2 originally occurred.

Since the paths have these beginning or ending steps, each path can be separated into three parts, which we denote P_1, P_2, and P_3, each of which could be empty. Consider the left Dyck path. P_1 is the path before the up step corresponding to the last down step of the Dyck path. P_1 does not include the guaranteed up and down steps. To find P_2, delete P_1 and its guaranteed up and down steps from the Dyck path. Then P_2 is found similarly to P_1 and does not include the guaranteed up and down steps as well. We then delete P_2 along with its guaranteed up and down steps from the Dyck path. Whatever is left of the Dyck path is P_3.

Now, consider the right Dyck path. Imagine a line that extends from the beginning vertex of the Dyck path to the end vertex of the Dyck path. We will call this line the horizon. P_1 is the path after the first down step that touches the horizon. Again, P_1 does not include the guaranteed up and down steps. Delete P_1 and its guaranteed up and down steps from the Dyck path. P_2 is found similarly to P_1. P_3 is whatever is left of the path after P_2 and its guaranteed up and down steps are deleted. We now have six Dyck paths that need to be combined in a way to form three Dyck paths.

To construct three Dyck paths from the six Dyck paths, we attach P_1 from the left Dyck path and P_1 from the right Dyck path, and similarly for P_2 and P_3. In order to attach each pair of Dyck paths together, the right Dyck path is elevated by an up step, which results in an extra down step at the end. We can see this process is reversible, and given three Dyck paths, we can first create six paths by identifying the last horizon-touching point, and recombine them to form the two paths in the inverse process to the one described here. Therefore, we have shown ϕ is a bijection, as desired, and we now have three Dyck paths to represent each σ. $\square$

Figure 4 shows an example of the process of representing a permutation in $\Phi_n^{i,j,k}$ as two Dyck paths. In this example, the top leftmost figure shows $\sigma = 865294173$ as an $n \times n$ array with the diagonal drawn in. We put σ_m on the squares (m, σ_m) for all elements σ_m in σ. This σ is a member of $\Phi_9^{2,4,7}$ where $\sigma_a = 2, \sigma_b = 4, \sigma_c = 7$, and $a = 4$, $b = 6$, and $c = 8$. The middle figure in this example shows where we place the rectangles, at $(\sigma_4, 4)$ and $(\sigma_6, 2)$. A path is drawn from (0, 9) to (9, 0) using only down steps and right steps. All the numbers and rectangles below the diagonal, except 2, remain to the right of the path, while the path remains as close to the diagonal as possible. One path ends at the first rectangle and extends to the diagonal, underneath 4. The next path then begins at the diagonal to the left of 4, and continues in the same way. After removing the grid, numbers, and rectangles, the two Dyck paths are separated and rotated 135° to obtain the two Dyck paths at the bottom of Figure 4.

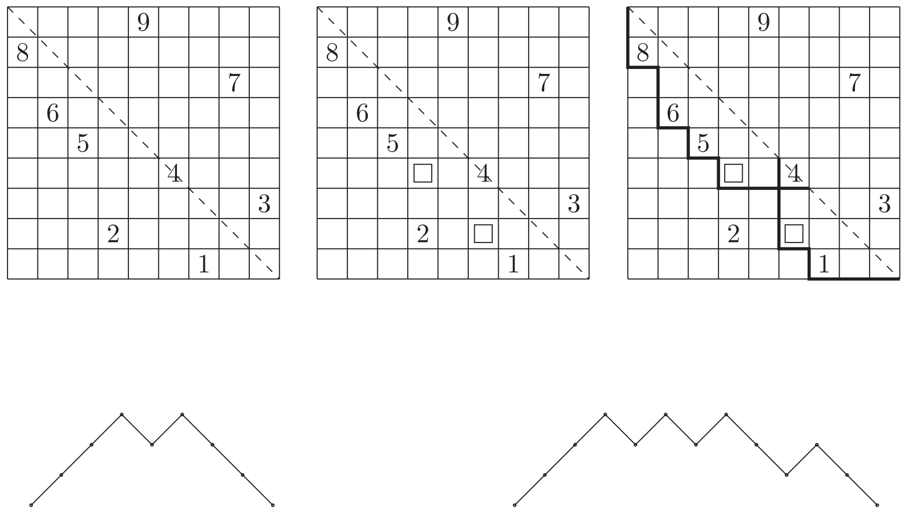

Figure 4: The process of representing a permutation as two Dyck paths.

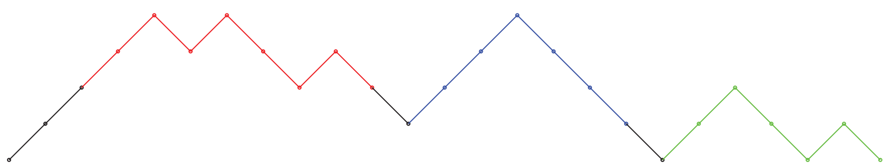

Figure 5: A Dyck path colored into its P_1, P_2, and P_3.

Consider the Dyck path in Figure 5. We can divide this Dyck path into three separate Dyck paths P_1, P_2, and P_3, as explained previously. Since this Dyck path begins in two up steps and does not end in two down steps, we know this must be the right Dyck path.

P_1 is the path after the first horizon-touching down step, which in Figure 5, is colored in green. After deleting all of the green path along with its guaranteed up and down steps, P_2 is the path after the first horizon-touching down step, as colored in blue. After deleting the blue path along with its guaranteed up and down steps, P_3 is what remains and is colored in red.

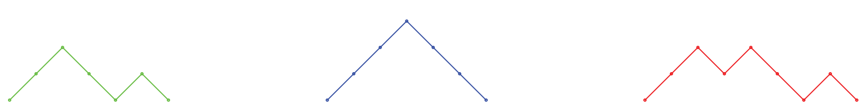

Figure 6: The three separate Dyck paths from Figure 5.

Figure 6 shows the resulting three Dyck paths. Doing this process on two Dyck paths gives six resulting Dyck paths.

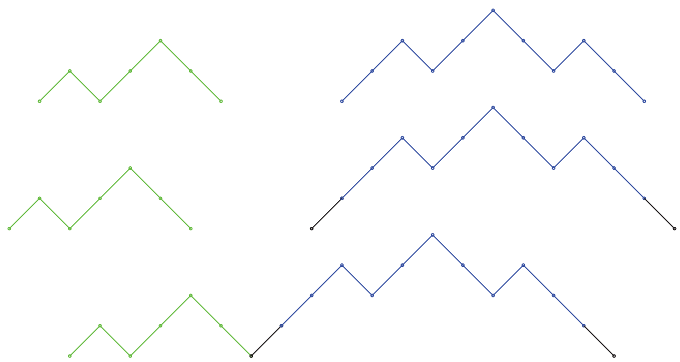

Figure 7: Combining two Dyck paths $UDUUDD$ and $UUDUUDDUDD$ to obtain $UDUUDDUUUDUUDDUDDD$.

In Figure 7 we show the process of gluing together two Dyck paths in which we elevate the second Dyck path by one up step, shown in black, then attach it to the first. We know this can be done by the first return decomposition of the Dyck paths.

In Figure 8, we continue our example of $\sigma = 865294173$, in which we started by dividing the permutation into two Dyck paths.

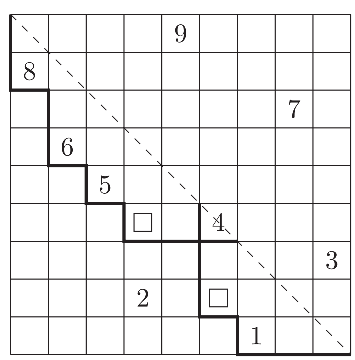

Figure 8: The permutation $\sigma = 865294173$ on an array.

In Figure 9 we rotate the two Dyck paths 135° clockwise and color it according to each Dyck path's P_1, P_2, and P_3. Notice both Dyck paths contain no green paths, meaning P_1 is the empty Dyck path for both Dyck paths. Also notice there is no blue path on the left Dyck path, meaning P_2 on the left is the empty Dyck path.

Figure 9: The two Dyck paths, colored, from the example in Figure 8 rotated 135° clockwise.

In Figure 10, we have the three Dyck paths to which this permutation now corresponds. We created these Dyck paths from taking each P_i on the left, adding an upstep, then adding the corresponding P_i on the right, and finally a downstep for $1 \leq i \leq 3$.

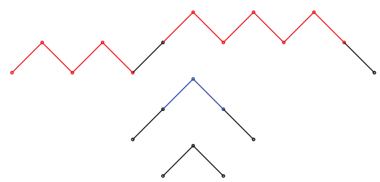

Figure 10: The resulting three Dyck paths.

It is well known that the number of Dyck paths containing n up steps corresponds to the nth Catalan number. For this example, we now see the permutation $\sigma = 865294173$ corresponds to Dyck paths of length 6, 2, and 1.

Since we are showing ϕ is a bijection, we will present an example now starting with three Dyck paths and construct the corresponding permutation in $\Phi_n^{i,j,k}$.

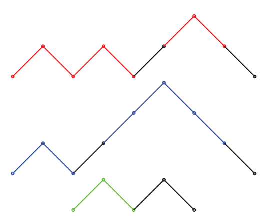

Figure 11: Our three beginning Dyck paths of length 4, 4, and 2, colored according to the six Dyck paths from which they are constructed.

Figure 11 depicts the three Dyck paths we will use in this example. The green, blue, and red paths before the first black up step in these three Dyck paths make up P_1, P_2, and P_3, respectively, of the left Dyck path. Similarly, the green, blue, and red paths after the first up step in these three Dyck paths make up P_1, P_2, and P_3, respectively, of the right Dyck path. Notice P_1 of the right Dyck path is empty.

Figure 12: The two Dyck paths created by combining paths P_1, P_2, and P_3 for the left and right Dyck paths.

When we combine P_1, P_2, and P_3 of each Dyck path, we obtain the two Dyck paths shown in Figure 12. Once we have these two Dyck paths, we can rotate them 135° counterclockwise, so the down steps correspond to straight down steps and the up steps correspond to right steps and place the paths on a 10×10 array, as shown in Figure 13. We then place the rectangles where they belong based on where the two Dyck paths cross on the array. This allows us to see $i = 4$ and $j = 6$. Based on how the Dyck path falls on the array, we can also right away place the numbers that fall below the diagonal by looking where a down step is immediately followed by a right step. From there, there is only one way to arrange the rest of the numbers so there is only one occurrence of a 123 and the Dyck path still represents the permutation. For this example, our resulting permutation is $\sigma = 8(10)49637251$ as shown in the figure on the right, which is a member of $\Phi_{10}^{4,6,7}$. We now see this permutation σ corresponds to Dyck paths of length 4, 4, and 2.

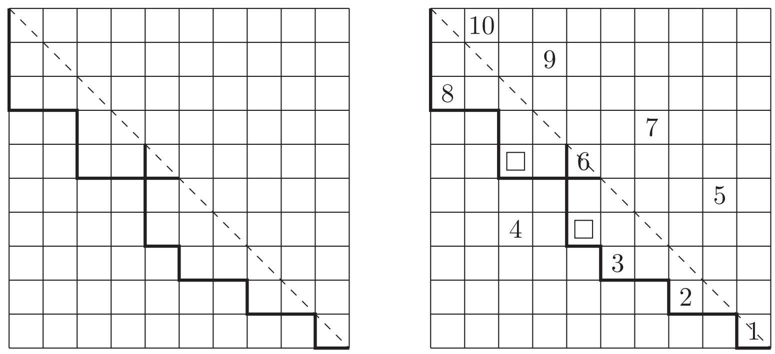

Figure 13: The two Dyck paths on 10×10 array

Corollary 15.

$$|\Phi_n| = \sum_{\substack{r,s,t\geq 1,\\ r+s+t=n}} C_r C_s C_t$$

This corollary can be more quickly proved using generating functions, but this bijective proof helps to illuminate the deep connections between pattern-avoiding and pattern-containing permutations and Dyck paths.

Acknowledgments

Special thanks to Dun Qui for his assistance with this bijection. Also thank you to the National Science Foundation DMS - 1451801 Grant for funding the program that enabled this research. Finally, thanks to the Office of Research and Sponsored Programs at the University of Wisconsin-Eau Claire for providing additional support for student research.

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