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Seven Dimensional Lie Algebras With Nilradical Isomorphic to $A_{5,1} \oplus \mathbb{R}$

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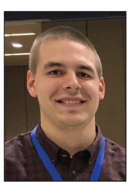

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Abstract This paper is the third in a series to classify the indecomposable solvable seven-dimensional Lie algebras with six-dimensional nilradical. We address the case where the nilradical is isomorphic to a direct sum of the first five-dimensional non-decomposable algebra and the one-dimensional algebra, denoted by $A_{5,1} \oplus \mathbb{R}$. We follow a technique that was first introduced by G. M. Mubarakzyanov.

1 Introduction

An n-dimensional real *Lie algebra* $\mathfrak{g}$ is an n-dimensional vector space over $\mathbb{R}$ with a skew-symmetric bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, (X, Y) \mapsto [X, Y]$, called the Lie bracket, satisfying the Jacobi identity. That is, the Lie bracket satisfies the following properties.

- For all $X, Y \in \mathfrak{g}, [X, Y] = -[Y, X]$.
- For all $a, b, c, d \in \mathbb{R}$ and $X, Y, Z, W \in \mathfrak{g}$,
$$[aX + bY, cZ + dW] = ac[X, Z] + ad[X, W] + bc[Y, Z] + bd[Y, W].$$
- For all $X, Y, Z \in \mathfrak{g}$,
$$\big[[X, Y], Z\big] + \big[[Y, Z], X\big] + \big[[Z, X], Y\big] = 0 \text{ (the Jacobi identity)}.$$

A familiar Lie algebra example is $\mathbb{R}^3$ with the cross product. The skew-symmetry and bilinear properties are naturally satisfied; the reader is encouraged to verify that the cross product on $\mathbb{R}^3$ satisfies the Jacobi identity.

For a Lie algebra $\mathfrak{g}$, we define the *derived series* $\mathfrak{g}^1, \mathfrak{g}^2, \ldots$ by setting $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = \{[X, Y] : X, Y \in \mathfrak{g}\}$ and, using the same notation for $i > 1$, $\mathfrak{g}^i = [\mathfrak{g}^{i-1}, \mathfrak{g}^{i-1}]$. If there is an integer N such that $\mathfrak{g}^N = 0$, we say that $\mathfrak{g}$ is a *solvable* algebra.

We also define the *lower central series* $\mathfrak{g}_1, \mathfrak{g}_2, \ldots$ by setting $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ and, for $i > 1$, $\mathfrak{g}_i = [\mathfrak{g}, \mathfrak{g}_{i-1}] = \{[X, Y] : X \in \mathfrak{g}, Y \in \mathfrak{g}_{i-1}\}$. If there is an integer N such that $\mathfrak{g}_N = 0$, we say that $\mathfrak{g}$ is a *nilpotent* algebra.

A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be an *ideal* of $\mathfrak{g}$ if and only if for every $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$, $[X, Y] \in \mathfrak{h}$. The maximal nilpotent ideal of $\mathfrak{g}$ is called the *nilradical* of $\mathfrak{g}$.

For the elementary theory of Lie algebras, refer to [4, 6, 7]. It has to be understood that classifying solvable Lie algebras is a different exercise from studying the semisimple algebras. The problem of classifying all semisimple Lie algebras over the field of complex numbers was solved by Cartan in 1894 [1]. Gantmacher then classified them over the field of real numbers 1939 [2]. For solvable indecomposable Lie algebras, the problem is much more difficult. The classification of solvable Lie algebras only exists for low dimensions and was performed by, among others, Mubarakzyanov for solvable Lie algebras of dimension $n \leq 5$ over the field of real numbers and partially over the field of complex numbers in [9] and [10]. Mubarakzyanov's results are summarized in [12]. Mubarakzyanov also considered dimension six and classified solvable Lie algebras with a codimension-one nilradical [11]. Shabanskaya and Thompson refined his results and found some missing cases in [15, 16]. Then Turkowski classified six-dimensional solvable Lie algebras with a codimension-two nilradical in [17]. Nilpotent Lie algebras in dimension six were studied as far back as Umlauf [18], and later by Morozov [8].

It is probably impossible to classify solvable Lie algebras in general in arbitrary dimension. Shabanskaya and Thompson solved the solvable extension problem of an arbitrary nilpotent algebra of dimension n into a solvable one of dimension $n+1$ or $n+2$ in [14]; our project is a special case of such an extension. We follow the solvable extension technique first introduced by Mubarakzyanov in [11].

The first step in classifying solvable Lie algebras in a specific dimension is to find the possible nilradicals. A general theorem asserts that if $\mathfrak{g}$ is an n-dimensional solvable Lie algebra, the dimension of its nilradical $\mathfrak{nil}(\mathfrak{g})$ is at least $\frac{n}{2}$ [11]. So for $n = 7$, the possibilities are that the nilradical is of dimension seven, six, five, or four. The seven-dimensional nilradicals, called the nilpotents, were studied by Seely over $\mathbb{R}$ [13] and by Gong over $\mathbb{C}$ [3]. The four-dimensional nilradical case was studied by Hindeleh and Thompson [5]. The six-dimensional nilradical case is a big project that we are currently working on, and this paper deals with the class where the nilradical is isomorphic to the direct sum of the first non-decomposable five-dimensional and the one-dimensional Lie algebra, denoted by $A_{5,1} \oplus \mathbb{R}$. The five-dimensional nilradical case is still an open problem.

Once a specific six-dimensional nilpotent algebra is chosen to be the nilradical for the seven-dimensional solvable algebra to be classified, one must make sure that this nilpotent algebra is an ideal inside the seven-dimensional one, and that the Jacobi identity is satisfied in the most general possible way. After that, one carries out the technique of 'absorption'; this method may make it possible to simplify the algebra.

In this project we are considering a seven-dimensional solvable indecomposable Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ that has a six-dimensional nilradical $\mathfrak{nil}(\mathfrak{g})$. In our case we use $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ as the basis of $\mathfrak{g}$, and $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ as the basis of $\mathfrak{nil}(\mathfrak{g})$. We begin by finding all possible nilradical ideals $\mathfrak{nil}(\mathfrak{g})$. Once we know $\mathfrak{nil}(\mathfrak{g})$, to obtain $\mathfrak{g}$, all that remains is to find $-ad(e_7)$ and simplify it where $ad(e_i)$ denotes the matrix corresponding to the basis vector e_i in the adjoint representation of $\mathfrak{g}$.

To perform absorption we make a change of basis of the form

$$e_i^{'} = e_i \,(1 \le i \le 6), \quad e_7^{'} = e_7 + \sum_{i=1}^{6} a^i e_i.$$

Then we obtain

$$\overline{ad}(e_7^{'}) = \overline{ad}(e_7) + \overline{ad}\left(\sum_{i=1}^{6} a^i e_i\right)$$

where $\overline{ad}$ denotes the restriction of the adjoint representation to $\mathfrak{nil}(\mathfrak{g})$. As such it may be possible to simplify $ad(e_7)$ by using e_i's which do not belong to the center of $\mathfrak{g}$ and then solve linear equations for the unknown a^i. Later on it may be necessary to repeat the absorption step after making further reductions in $ad(e_7)$. In that case it is important to know that such an absorption will not affect other entries in $ad(e_7)$ possibly already normalized.

After absorption the work really begins. We have to try to remove and simplify entries in $ad(e_7)$. We find that these further reductions necessarily break into many subcases depending on whether the entries in $ad(e_7)$ satisfy certain inequalities.

The conditions introduced on parameters are necessary to make sure that the algebras remain solvable and indecomposable. Finally, we verify that the algebras found are non-isomorphic by checking different values for the parameters. We can confidently claim that we completed the classification problem for the seven-dimensional solvable algebras with six-dimensional nilradical isomorphic to $A_{5,1} \oplus \mathbb{R}$.

Throughout our paper P denotes a change of basis matrix applied to an initial Lie algebra which gives a new form of the algebra. In order to shorten the explanation and avoid excessive verbiage we will usually just use a word like 'hence' to indicate that a new form of the algebra has been found.

In Section 2, we outline all cases for the classification problem of the seven-dimensional solvable algebras with six-dimensional nilradical. In Section 3, we focus on the case where the six-dimensional nilradical is isomorphic to $A_{5,1} \oplus \mathbb{R}$ and reduce the problem into lifting twelve possible five-dimensional algebras, namely $A_{5,7} - A_{5,18}$ from Patera's list [12]. In Section 4, we examine each one of the algebras $A_{5,7} - A_{5,18}$ and try lifting them to our desired seven-dimensional algebra.

2 Nilradicals

Fix a dimension $n \in \mathbb{N}$. Let e_i $(1 \leq i \leq n)$ be a basis for a finite n-dimensional Lie algebra defined by the bracket relations $[e_i, e_j] = \sum_{k=1}^{n} C_{ij}^{k} e_k$, where C_{ij}^{k} are the structure constants of the Lie algebra and form a three-dimensional array. Let us find a Lie algebra isomorphic to the existing one in a new basis $\{e'_i\}$, $1 \leq i \leq n$, knowing that $e'_i = \sum_{j=1}^{n} \left(P^T\right)_i^j \cdot e_j$ and $e_i = \sum_{j=1}^{n} \left[\left(P^T\right)^{-1}\right]_i^j \cdot e'_j$, where i is a fixed column, and P is the change of basis matrix which connects the vectors from the two different bases. Then

$$\begin{aligned}
[e'_i, e'_j] &= \left[\sum_{k=1}^{n} \left(P^T\right)_i^k \cdot e_k, \sum_{m=1}^{n} \left(P^T\right)_j^m \cdot e_m\right] \\
&= \sum_{k=1}^{n} \sum_{m=1}^{n} \left(P^T\right)_i^k \cdot \left(P^T\right)_j^m \cdot [e_k, e_m] \\
&= \sum_{r=1}^{n} \sum_{k=1}^{n} \sum_{m=1}^{n} \left(P^T\right)_i^k \cdot \left(P^T\right)_j^m \cdot C_{km}^r e_r \\
&= \sum_{s=1}^{n} \sum_{r=1}^{n} \sum_{k=1}^{n} \sum_{m=1}^{n} \left(P^T\right)_i^k \cdot \left(P^T\right)_j^m \cdot C_{km}^r \cdot \left[\left(P^T\right)^{-1}\right]_r^s \cdot e'_s \\
&= \sum_{s=1}^{n} \left(\sum_{r=1}^{n} \sum_{k=1}^{n} \sum_{m=1}^{n} \left(P^T\right)_i^k \cdot \left(P^T\right)_j^m \cdot C_{km}^r \cdot \left[\left(P^T\right)^{-1}\right]_r^s\right) \cdot e'_s.
\end{aligned}$$

The Lie algebra in the new basis is given by $[e'_i, e'_j] = \sum\limits_{s=1}^{n} C'^{s}_{ij} e'_s$. Hence,

$$C'^{s}_{ij} = \sum_{r=1}^{n} \sum_{k=1}^{n} \sum_{m=1}^{n} \left(P^T\right)_i^k \cdot \left(P^T\right)_j^m \cdot C_{km}^r \cdot \left[\left(P^T\right)^{-1}\right]_r^s.$$

The following Lie algebras are all the nilpotent Lie algebras up to isomorphism in dimension six and are all the possible nilradicals for a seven-dimensional solvable indecomposable Lie algebra with a co-dimension one nilradical.

i. $\mathbb{R}^6$,

ii. $H \oplus \mathbb{R}^3$,

iii. $A_{4,1} \oplus \mathbb{R}^2$,

iv. $A_{5,1} \oplus \mathbb{R} - A_{5,6} \oplus \mathbb{R}$ (six cases),

v. $A_{6,1} - A_{6,22}$ (22 cases),

vi. $H \oplus H$,

where $\mathbb{R}^n$ denotes the n−dimensional abelian algebra, H denotes the three-dimensional Heisenberg algebra, and $A_{n,k}$ denotes the k^{th} algebra of dimension n from Patera's list. The first two cases are treated in work currently submitted for publication. The focus of this article is part of the fourth case where the nilradical is isomorphic to $A_{5,1} \oplus \mathbb{R}$. We completed multiple other cases which we intend to prepare for publication.

3 Satisfying the Jacobi identity

The non-zero brackets in the nilradical ($A_{5,1} \oplus \mathbb{R}$) are $[e_3, e_5] = e_1$ and $[e_4, e_5] = e_2$, where $\mathbb{R}$ is spanned by e_6. We prefer to permute the basis elements using the change of basis $e'_1 = e_1, e'_2 = e_2, e'_3 = e_6, e'_4 = e_4, e'_5 = e_3, e'_6 = e_5$, and then rename our basis elements $e'_i = e_i$. Therefore, the nilradical non-zero bracket relations are $[e_4, e_6] = e_2$ and $[e_5, e_6] = e_1$. Hence the non-zero brackets in our seven-dimensional algebra are

$$[e_4, e_6] = e_2, [e_5, e_6] = e_1, [e_i, e_7] = \sum_{k=1}^{6} b_i^k e_k,$$

where $1 \le i \le 6$.

Notice that the adjoint representation for e_7, denoted by $ad(e_7)$, is the matrix $[b_i^k](i, k = 1 \dots 7)$, where the b_i^k are defined by $[e_7, e_i] = \sum_{k=1}^{7} b_i^k e_k$. Since $[e_7, e_7] = 0$, the last column of $ad(e_7)$ is the zero vector. Since $\{e_1, \dots, e_6\}$ spans an ideal, $b_i^7 = 0$ for $i = 1, \dots, 7$. Thus the last row of $ad(e_7)$ contains zeros. Let A be the matrix obtained from the adjoint representation $-ad(e_7)$ by deleting the last column and row. Finding all solutions for the matrix A will solve the classification problem for this class.

Next, we impose the Jacobi identity conditions on $\{b_i^k\}$ to make sure that we have a Lie algebra. Notice that the Jacobi identity is satisfied for all triples $\{e_i, e_j, e_k\}$ where $1 \le i, j, k \le 6$. Hence, it suffices to check triples $\{e_i, e_j, e_7\}$ where $1 \le i < j \le 6$. For example, checking the Jacobi identity for $\{e_1, e_4, e_7\}$

requires

$$\begin{aligned}
\big[[e_1,e_4],e_7\big]+\big[[e_4,e_7],e_1\big]+\big[[e_7,e_1],e_4\big] &= 0\\
\big[0,e_7\big]+\sum_{k=1}^{6} b_4^k\big[e_k,e_1\big]-\sum_{k=1}^{6} b_1^k\big[e_k,e_4\big] &= 0\\
0+0+b_1^6 e_2 &= 0,
\end{aligned}$$

from which we conclude that $b_1^6 = 0$. We use a routine we wrote on Maple® to check the rest of the Jacobi conditions. We note that because of the Jacobi identity, A must be of the form

$$A = \begin{bmatrix} b_5^5+b_6^6 & b_2^1 & b_3^1 & b_4^1 & b_5^1 & b_6^1\\ b_1^2 & b_6^6+b_4^4 & b_3^2 & b_4^2 & b_5^2 & b_6^2\\ 0 & 0 & b_3^3 & b_4^3 & b_5^3 & b_6^3\\ 0 & 0 & 0 & b_4^4 & b_1^2 & b_6^4\\ 0 & 0 & 0 & b_2^1 & b_5^5 & b_6^5\\ 0 & 0 & 0 & 0 & 0 & b_6^6 \end{bmatrix}.$$

This form of $-ad(e_7)$ restricted to the nilradical is the most general form for the Jacobi identity to hold given that the nilradical is isomorphic to $A_{5,1}\oplus\mathbb{R}$. Now we apply the technique of absorption. We consider the change of basis that fixes $e_1,\ldots,e_6$ and changes e_7 to $e_7' = e_7 + b_6^2\, e_4 + b_6^1\, e_5 - b_5^1\, e_6$. This change of basis turns A into

$$A' = \begin{bmatrix} b_5^5+b_6^6 & b_2^1 & b_3^1 & b_4^1 & 0 & 0\\ b_1^2 & b_6^6+b_4^4 & b_3^2 & b_4^2 & b_5^2 & 0\\ 0 & 0 & b_3^3 & b_4^3 & b_5^3 & b_6^3\\ 0 & 0 & 0 & b_4^4 & b_1^2 & b_6^4\\ 0 & 0 & 0 & b_2^1 & b_5^5 & b_6^5\\ 0 & 0 & 0 & 0 & 0 & b_6^6 \end{bmatrix}. \tag{1}$$

The algebra bracket relations are now reduced to

$$
\begin{aligned}
&[e_4, e_6] = e_2,\\
&[e_5, e_6] = e_1,\\
&[e_1, e_7] = (b_6^6 + b_5^5)e_1 + b_1^2 e_2,\\
&[e_2, e_7] = b_2^1 e_1 + (b_6^6 + b_4^4)e_2,\\
&[e_3, e_7] = b_3^1 e_1 + b_3^2 e_2 + b_3^3 e_3,\\
&[e_4, e_7] = b_4^1 e_1 + b_4^2 e_2 + b_4^3 e_3 + b_4^4 e_4 + b_2^1 e_5,\\
&[e_5, e_7] = b_5^2 e_2 + b_5^3 e_3 + b_1^2 e_4 + b_5^5 e_5,\\
&[e_6, e_7] = b_6^3 e_3 + b_6^4 e_4 + b_6^5 e_5 + b_6^6 e_6.
\end{aligned} \tag{2}
$$

Notice that by (2), $\{e_1, e_2\}$ spans a two-dimensional ideal in the Lie algebra. Modding by $\{e_1, e_2\}$ will reduce the algebra to a five-dimensional algebra with brackets

$$
\begin{aligned}
&[e_3, e_7] = b_3^3 e_3,\\
&[e_4, e_7] = b_4^3 e_3 + b_4^4 e_4 + b_2^1 e_5,\\
&[e_5, e_7] = b_5^3 e_3 + b_1^2 e_4 + b_5^5 e_5,\\
&[e_6, e_7] = b_6^3 e_3 + b_6^4 e_4 + b_6^5 e_5 + b_6^6 e_6.
\end{aligned} \tag{3}
$$

The algebra with non-zero bracket relations in (3) is isomorphic to a five-dimensional solvable algebra with a four-dimensional abelian nilradical. Hence it must be isomorphic to one of the $A_{5,7} - A_{5,19}$ algebras from Patera's list. We next examine each one of those twleve cases to determine the lower right 4×4 block of A'. Since the parameters of the upper 2×2 block occur in the lower 4×4 block, the upper left 2×2 block of A' in (1) is determined as well.

After that, we will try get rid of the remaining parameters ($b_3^1, b_3^2, b_4^1, b_4^2$, and b_5^2) via change of bases of the form $e_i' = e_i$ where $i \in \{1, 2\}$, and $e_i' = x_i^1 e_1 + x_i^2 e_2 + e_i$ where $i \in \{3, 4, 5, 6\}$. Since e_1 and e_2 are in the center of the nilradical, this change of basis preserves the bracket relations on the nilradical. In each case we provide the appropriate change of basis. In the cases where we are not able to eliminate the parameters $b_3^1, b_3^2, b_4^1, b_4^2$, and b_5^2, we replace them with the Greek letters $\alpha, \beta, \gamma, \delta$, and ϵ respectively, in order to distinguish them from parameters inherited from the five-dimensional algebras.

Notice that the choice we made in permuting our basis elements earlier was intended to align our bases after modding by $\{e_1, e_2\}$ with the five-dimensional algebras $A_{5,7} - A_{5,19}$ from Patera's list.

4 Finding seven dimensional Lie algebras

As promised above, we now consider in turn each of the algebras $A_{5,7} - A_{5,18}$ from [12] and see if we can extend to a seven-dimensional algebra. We list the non-zero bracket relations for each one of those algebras below for convenience. We show detailed work for the first case, and proceed similarly for the other eleven cases, listing final results in Table 1.

(i) $A_{5,7} : [e_1, e_5] = e_1, [e_2, e_5] = ae_2, [e_3, e_5] = be_3, [e_4, e_5] = ce_4, (abc \neq 0, -1 \leq c \leq b \leq a \leq 1)$.

Hence $-ad(e_7)$ restricted to the nilradical is

$$A' = \begin{bmatrix} b+c & 0 & b_3^1 & b_4^1 & 0 & 0 \\ 0 & a+c & b_3^2 & b_4^2 & b_5^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{bmatrix}.$$

We are only able to get rid of b_4^2 with the change of basis that fixes all of them except e_4, where

$$e_4' = -\frac{b_4^2}{c} e_2 + e_4.$$

For the rest of the paper, we replace the upper right parameters with greek letters to distinguish them from inherited parameters. Therefore we get the algebra $g_{7,47}$:

$$\begin{aligned} &[e_4, e_6] = e_2, [e_5, e_6] = e_1, \\ &[e_1, e_7] = (b+c)e_1, [e_2, e_7] = (a+c)e_2, [e_3, e_7] = \alpha e_1 + \beta e_2 + e_3, \\ &[e_4, e_7] = \gamma e_1 + ae_4, [e_5, e_7] = \epsilon e_2 + be_5, [e_6, e_7] = ce_6. \end{aligned}$$

(ii) $A_{5,8} : [e_2, e_5] = e_1, [e_3, e_5] = e_3, [e_4, e_5] = ce_4, (0 < |c| \leq 1)$.

We are only able to get rid of b_3^2 and b_4^2 with the change of basis that fixes all of them except e_3 and e_4, where

$$\begin{aligned} e_3' &= -\frac{b_3^2}{c} e_2 + e_3, \\ e_4' &= -\frac{b_4^2 c + b_3^2}{c^2} e_2 + e_4. \end{aligned}$$

Therefore we get the algebra $g_{7,48}$, which is listed in Table 1.

(iii) $A_{5,9} : [e_1, e_5] = e_1, [e_2, e_5] = e_1 + e_2, [e_3, e_5] = be_3, [e_4, e_5] = ce_4, (0 \neq c \leq b)$.

We are only able to get rid of b_3^2 and b_4^2 with the change of basis that fixes all except e_3 and e_4, where

$$\begin{aligned} e_3' &= -\frac{b_3^2}{c} e_2 + e_3, \\ e_4' &= -\frac{b_4^2 c + b_3^2}{c^2} e_2 + e_4. \end{aligned}$$

Therefore we get the algebra $g_{7,49}$, which is listed in Table 1.

(iv) $A_{5,10} : [e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_4$.
We are able to get rid of $b_3^1, b_4^1, b_3^2, b_4^2$ and b_5^2 with the change of basis that fixes all except e_3, e_4 and e_5, where

$$\begin{aligned}
e_3' &= -b_3^1 e_1 + (-b_3^2 + b_3^1) e_2 + e_3, \\
e_4' &= -(b_4^1 + b_3^1) e_1 + (-b_4^2 + b_4^1 + 2b_3^1 - b_3^2) e_2 + e_4, \\
e_5' &= (-b_4^2 - b_5^2 + b_4^1 + 2b_3^1 - b_3^2) e_2 + e_5.
\end{aligned}$$

Therefore we get the algebra $g_{7,50}$, which is listed in Table 1.

(v) $A_{5,11} : [e_1, e_5] = e_1, [e_2, e_5] = e_1 + e_2, [e_3, e_5] = e_2 + e_3, [e_4, e_6] = ce_4, (c \neq 0)$.
We are able to get rid of $b_3^1, b_4^1, b_3^2, b_4^2$ and b_5^2 with the change of basis that fixes all except e_3, e_4 and e_5, where

$$\begin{aligned}
e_3' &= -\frac{b_3^1}{c} e_1 + \left(\frac{-b_3^2\, c + b_3^1}{c^2} \right) e_2 + e_3, \\
e_4' &= -\left(\frac{b_4^1\, c + b_3^1}{c^2} \right) e_1 + \left(\frac{-b_4^2\, c^2 - b_3^2\, c + b_4^1\, c + 2\, b_3^1}{c^3} \right) e_2 + e_4, \\
e_5' &= \left(\frac{-b_5^2\, c^3 - b_4^2\, c^2 - b_3^2\, c + b_4^1\, c + 2\, b_3^1}{c^4} \right) e_2 + e_5.
\end{aligned}$$

Therefore we get the algebra $g_{7,51}$, which is listed in Table 1.

(vi) $A_{5,12} : [e_1, e_5] = e_1, [e_2, e_5] = e_1 + e_2, [e_3, e_5] = e_2 + e_3, [e_4, e_6] = e_3 + e_4$.
We are able to get rid of $b_3^1, b_4^1, b_3^2, b_4^2$ and b_5^2 with the change of basis that fixes all of them except e_3, e_4, e_5 and e_6, where

$$\begin{aligned}
e_3' &= -b_3^1\, e_1 + \left(-b_3^2 + b_3^1\right) e_2 + e_3, \\
e_4' &= \left(-b_4^1 - b_3^1\right) e_1 + \left(-b_4^2 + b_4^1 + 2b_3^1 - b_3^2\right) e_2 + e_4, \\
e_5' &= \left(-b_4^1 - b_3^1\right) e_1 + \left(-b_4^2 - b_5^2 + 2b_4^1 + 3b_3^1 - b_3^2\right) e_2 + e_5, \\
e_6' &= \left(-b_4^1 - b_3^1\right) e_1 + \left(3b_4^1 + 4b_3^1 - b_4^2 - b_5^2 - b_3^2\right) e_2 + e_6.
\end{aligned}$$

Therefore we get the algebra $g_{7,52}$, which is listed in Table 1.

(vii) $A_{5,13} : [e_1, e_5] = e_1, [e_2, e_5] = a\, e_2, [e_3, e_5] = p\, e_3 - q\, e_4, [e_4, e_6] = q\, e_3 + p\, e_4, (ap \neq 0, |a| \leq 1)$.
Notice that q must equal 0 for the algebra to satisfy the Jacobi identity. We are only able to get rid of b_4^2 and b_5^2 with the change of basis that fixes all of them except e_4 and e_5, where

$$\begin{aligned}
e_4' &= -\frac{b_4^2}{p} e_2 + e_4, \\
e_5' &= -\frac{b_5^2}{a} e_2 + e_5.
\end{aligned}$$

Therefore we get the algebra $g_{7,53}$, which is listed in Table 1.

(viii) $A_{5,14} : [e_1, e_5] = e_1, [e_3, e_5] = pe_3 - e_4, [e_4, e_5] = e_3 + pe_4$.

This algebra does not have the same form of $-ad(e_7)$ in (1). Hence, this case does not give us a corresponding seven-dimensional algebra.

(ix) $A_{5,15} : [e_1, e_5] = e_1, [e_2, e_5] = e_1 + e_2, [e_3, e_5] = ae_3, [e_4, e_5] = e_3 + ae_4$, $(|a| \le 1)$.

Hence $-ad(e_7)$ restricted to the nilradical is

$$A' = \begin{bmatrix} 2a & 0 & b_3^1 & b_4^1 & 0 & 0 \\ 0 & 1+a & b_3^2 & b_4^2 & b_5^2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 1 \\ 0 & 0 & 0 & 0 & 0 & a \end{bmatrix}.$$

We are not able to get rid of any of the parameters in this case. Therefore we get the algebra $g_{7,54}$, which is listed in Table 1, where

$$\begin{aligned} &[e_4, e_6] = e_2, [e_5, e_6] = e_1, \\ &[e_1, e_7] = 2ae_1, [e_2, e_7] = (a+1)e_2, [e_3, e_7] = \alpha e_1 + \beta e_2 + e_3, \\ &[e_4, e_7] = \gamma e_1 + \delta e_2 + e_3 + e_4, [e_5, e_7] = \epsilon e_2 + ae_5, [e_6, e_7] = e_5 + ae_6. \end{aligned}$$

(x) $A_{5,16} : [e_1, e_5] = e_1, [e_2, e_5] = e_1 + e_2, [e_3, e_5] = pe_3 - qe_4, [e_4, e_5] = qe_3 + pe_4, (q \neq 0)$.

This algebra does not have the same form of $-ad(e_7)$ in (1). Hence, this case does not give us a corresponding seven-dimensional algebra.

(xi) $A_{5,17} : [e_1, e_5] = pe_1 - e_2, [e_2, e_5] = e_1 + pe_2, [e_3, e_5] = qe_3 - se_4, [e_4, e_5] = se_3 + qe_4, (s \neq 0)$.

This algebra does not have the same form of $-ad(e_7)$ in (1). Hence, this case does not give us a corresponding seven-dimensional algebra.

(xii) $A_{5,18} : [e_1, e_5] = pe_1 - e_2, [e_2, e_5] = e_1 + pe_2, [e_3, e_5] = e_1 + pe_3 - e_4, [e_4, e_5] = e_2 + e_3 + pe_4, (p \ge 0)$.

This algebra does not have the same form of $-ad(e_7)$ in (1). Hence, this case does not give us a corresponding seven-dimensional algebra.

5 Conclusion

This completes the classification for the seven-dimensional indecomposable Lie algebras with a six-dimensional nilradical isomorphic to $A_{5,1} \oplus \mathbb{R}$. The restrictions on the parameters a, b, c and p are inherited from the algebras $g_{5,7} - g_{5,18}$, and are necessary to guarantee that those algebras remain unique within their class. We checked with Maple® that no special values for the parameters $\alpha, \beta, \gamma, \delta$ and ϵ will make two of the newly classified algebras isomorphic, or any of those algebras decomposable or nilpotent. We note that we choose to number the algebras in the appendix starting from 47 in order to preserve the sequence numbering with previous work. Substantial work was done for the case where the nilradical is isomorphic to $A_{5,2} \oplus \mathbb{R}$, and will be submitted soon.

Type	$[e_4, e_6]$	$[e_5, e_6]$	$[e_1, e_7]$	$[e_2, e_7]$	$[e_3, e_7]$	$[e_4, e_7]$	$[e_5, e_7]$	$[e_6, e_7]$	Conditions on Variables
$\mathfrak{g}_{7,47}$	e_2	e_1	$(b+c)e_1$	$(a+c)e_2$	$\alpha e_1 + \beta e_2 + e_3$	$\gamma e_1 + a e_4$	$\epsilon e_2 + be_5$	$c e_6$	$abc \neq 0$, $-1 \leq c \leq b \leq a \leq 1$
$\mathfrak{g}_{7,48}$	e_2	e_1	$(1+c)e_1$	$c e_2$	αe_1	$\gamma e_1 + e_3$	$\epsilon e_2 + e_5$	$c e_6$	$0 < \lvert c \rvert \leq 1$
$\mathfrak{g}_{7,49}$	e_2	e_1	$(b+c)e_1$	$(1+c)e_2$	$\alpha e_1 + e_3$	$\gamma e_1 + e_3 + e_4$	$\epsilon e_2 + be_5$	$c e_6$	$0 \neq c \leq b$
$\mathfrak{g}_{7,50}$	e_2	e_1	$e_1 + e_2$	e_2	0	e_3	e_4	e_6	
$\mathfrak{g}_{7,51}$	e_2	e_1	$(1+c) e_1 + e_2$	$(1+c) e_2$	e_3	$e_3 + e_4$	$e_4 + e_5$	ce_6	$c \neq 0$
$\mathfrak{g}_{7,52}$	e_2	e_1	$2 e_1 + e_2$	$2 e_2$	e_3	$e_3 + e_4$	$e_4 + e_5$	$e_5 + e_6$	
$\mathfrak{g}_{7,53}$	e_2	e_1	$2p e_1$	$(a+p) e_2$	$\alpha e_1 + \beta e_2 + e_3$	$\gamma e_1 + a e_4$	$p e_5$	$p e_6$	$ap \neq 0, \lvert a \rvert \leq 1$
$\mathfrak{g}_{7,54}$	e_2	e_1	$2a e_1$	$(a+1) e_2$	$\alpha e_1 + \beta e_2 + e_3$	$\gamma e_1 + \delta e_2 + e_3 + e_4$	$\epsilon e_2 + ae_5$	$e_5 + a e_6$	$\lvert a \rvert \leq 1$

Table 1: Seven-Dimensional Lie Algebras with Nilradical isomorphic to $A_{5,1} \oplus \mathbb{R} : [e_4, e_6] = e_2, [e_5, e_6] = e_1$.

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