The Banach-Tarski Paradox

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A paradox is, generally speaking, a disproof of our *intuitive* sense of what "should" be true. Through the ages, then, there have been many instructive and interesting paradoxes that have informed and reshaped mathematical intuition. As the field of mathematics has become more rigorous and less naïve, the consequences of this increasing emphasis on logical precision have often escaped mathematicians.

In this article, we will look at two famous paradoxes: Russell's Paradox and the paradox of Banach and Tarski. These are two of the realizations that have shaped the world of mathematics in which we live and work: it is a weirder one than the layman might realize.

Russell's Paradox

Until the mid-1800s, mathematicians approached set theory somewhat naïvely. They simply assumed that sets existed and that they were nice to deal with. Unfortunately, this didn't last: Bertrand Russell discovered his famous paradox. It disproved the intuitive idea of sets being simply any collection of objects. For suppose a set really were any collection of objects. Then, since sets themselves are objects, we could gather all sets into a set. What would happen if this set of all sets, call it A, actually existed? This monster would be quite difficult to imagine, not the least because it would be an element of itself: $A \in A$.

Let us concern ourselves with the following subset (and element) of A, the set $N = \{K | K \notin K\} \subseteq A$. Then is $N \in N$? Of course not: for if it were, then it could not be an element of itself by its very definition. But if it is not, then it exactly fits the criterion for inclusion back in itself! This is the paradox: the statement $N \in N$ should either be true or false—provided we assume that A exists, that is. We therefore have reached a contradiction and conclude that there can be no set of all sets.

This dealt quite an ugly blow to naïve set theory. Mathematics would no longer be able to take for granted its foundational assumptions: they would have to be reformulated and shown consistent, inasmuch as that is possible.

The Banach-Tarski Paradox

In the years following, some mathematicians formulated axioms for set theory which regulate what constitutes a set. One of these, the Axiom of Choice, raised a few eyebrows. The Axiom of Choice is fundamental to almost all modern mathematics. For example, it is equivalent to the Well-Ordering Principle (that every set can be well-ordered) and the assertion that an arbitrary cartesian product of nonempty sets is nonempty. The axiom simply states this:

Axiom of Choice. *Given any collection S of nonempty disjoint sets, there exists a set C which contains exactly one element from each element of S.*

The next little surprise we are going to examine is one of the ugly children of the Choice Axiom. It is known as the Banach-Tarski Paradox. Essentially, the theorem, published by S. Banach and A. Tarski in 1924 [1], shows that it is possible to take a solid ball in three-dimensional Euclidean space, partition it into a finite number of pieces, and then perform a finite number of rigid motions on them to rearrange the pieces into *two* balls, each of whose volume is equal to that of the first. From this, they concluded:

The Banach-Tarski Paradox. *Let A and B be bounded subsets of a Euclidean space in at least three dimensions, neither of which has an empty interior. Then there exist partitions of A and B into a finite number of disjoint subsets $A = A_1 \cup A_2 \cup \cdots \cup A_n$, $B = B_1 \cup B_2 \cup \cdots \cup B_n$, such that, for each i between 1 and n, A_i is congruent to B_i.*

This paradox implies the startling "pea and the sun" result:

A ball the size of a pea can be sliced up into a finite number of pieces, and those pieces moved by only rigid motions in three-dimensional space, to form a ball the size of the sun.

More generally, the same is true of any two bounded subsets of $\mathbb{R}^n$ ($n \geqslant 3$) with nonempty interiors. We are going to take a look at the proofs of the three-dimensional versions of these rather counterintuitive results. Before we do that, we need a few preliminaries.

Definition. *Let $A, B \subseteq \mathbb{R}^n$. A is said to be piecewise congruent to B, denoted by $A \approx B$, if A can be partitioned into finitely many pairwise disjoint subsets $A_1, A_2, A_3, \ldots, A_s$ such that each A_i can be translated and rotated to some B_i so as to form pairwise disjoint sets $B_1, B_2, B_3, \ldots, B_s$ whose union equals B. If $A \approx B' \subseteq B$, we write $A \preccurlyeq B$.*

Observe that this relation is transitive: if $A \preccurlyeq B$ and $B \preccurlyeq C$, then $A \preccurlyeq C$. Figure 1 depicts some piecewise congruent sets in the Euclidean plane. The reader should be advised, however, that in general no assumption is made about either the connectivity or the measurability of the individual pieces.

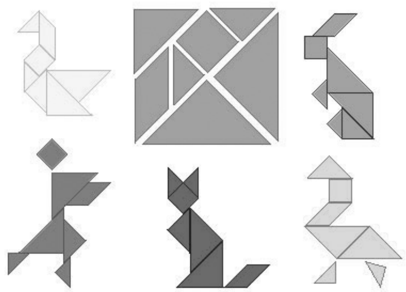

Figure 1: In this example from the children's game *Tangram*, all of the depicted shapes are piecewise congruent to the central square.

We need the following lemma:

Lemma. *If $A \preccurlyeq B$ and $B \preccurlyeq A$, then $A \approx B$.*

Proof. Say $A \approx B' \subseteq B$ and $B \approx A' \subseteq A$, and let $f : A \to B'$ and $g : B \to A'$ be the functions following the stipulated rigid motions. Put $C_0 = A - A'$ and $C_1 = g(f(C_0))$. Continue inductively by defining $C_{n+1} = g(f(C_n))$.

Put $C = \bigcup_{n=0}^{\infty} C_n$. Then $A - C \subseteq A' = A - C_0$ and $g^{-1}(A-C) = B - f(C)$, since $g(b) \in C$ if and only if $b \in f(C)$. Therefore, as $g^{-1} : A' \to B$ follows the motion for $A' \approx B$, we have $A - C \approx B - f(C)$. Moreover, $C \approx f(C)$, so $A \approx B$. □

Now we are ready for the main theorem:

Theorem (Banach-Tarski)**.** *Let $P \subseteq \mathbb{R}^3$ be a solid ball of any radius r. Then there exist subsets $A,\ B \subseteq P$ such that $A \cap B = \emptyset$ and $A \approx P \approx B$.*

Proof. We sketch a proof of this theorem in several steps. First, we prove it for the surface of P, then extend it to the solid ball. Specifically, let S be the surface of P, centered at the origin. We will first show that there are $A, B \subseteq S$ with $A \cap B = \emptyset$ and $A \approx S \approx B$, using only rotations about the origin.

Let ρ and σ be counterclockwise rotations about the x-axis and z-axis, respectively, each by the angle $\alpha = \arccos\left(\frac{1}{3}\right)$. Consider the set $G = \langle \sigma, \rho \rangle$ of all possible finite sequences of rotations $\sigma, \sigma^{-1}, \rho$ and ρ^{-1}, without trivial cancellation. Note that each sequence, under function composition, results in an overall rotation of S about some axis through the origin. Using elementary linear algebra, it can be shown that if the first of n moves in such a sequence is a z-axis rotation, then it transforms the unit vector $(1, 0, 0)$ into some vector of the form $\left(\frac{a}{3^n}, \frac{b\sqrt{2}}{3^n}, \frac{c}{3^n}\right)$ with $a, b, c \in \mathbb{Z}$ and $b \neq 0$. (See [3] for details.) Therefore, the empty sequence is the only sequence in G which equals the identity function, and different sequences in G result in different rotations.

Now, partition G into five disjoint sets: let X^+ be the set of all sequences whose last move is ρ, let X^- be the set of all sequences whose last move is ρ^{-1}, let Z^+ be the set of all sequences whose last move is σ, and let Z^- be the set of all sequences whose last move is σ^{-1}. Then $G = X^+ \cup X^- \cup Z^+ \cup Z^- \cup \{id\}$. Note that $G = X^+ \cup \rho X^-$ and $G = Z^+ \cup \sigma Z^-$.

Each rotation in $G - \{id\}$ fixes an axis through P. Let F be the set of points in S where these axes meet S. Because G is countable, F is countable. No element of $G - \{id\}$ fixes a point on $S - F$. Moreover, the orbit of any point of $S - F$ under G is a set of distinct points corresponding bijectively to G. Let M be a collection of exactly one point from the orbit of each point in $S - F$. Note that in order to create M, we must use the Axiom of Choice to choose exactly one point from each of the disjoint orbits of the points of $S - F$. Now define $A_1 = \{g(M) \mid g \in X^+\}$, $A_2 = \{g(M) \mid g \in X^-\}$, $B_1 = \{g(M) \mid g \in Z^+\}$, and $B_2 = \{g(M) \mid g \in Z^-\}$.

But now we have $A_1 \cup \rho(A_2) = S - F$ and $B_1 \cup \sigma(B_2) = S - F$. This is the heart of the proof: translating into the geometry of S the seemingly innocuous fact that G has pairs of disjoint subsets with the property that the second set of each pair can be shifted onto the complement of the first.

Next, let l be an axis that passes through the center of S missing F. Further, let τ be a rotation about l by an angle θ such that $\tau^n(F) \cap F = \emptyset$ for all $n > 0$ (recall that F is countable). We then have that $\tau^n(F) \cap \tau^m(F) = \emptyset$ for $0 \leqslant m < n$, since otherwise we would be able to spin the other way m times and contradict the assumption about θ. Put $\overline{F} = \cup_{n=0}^{\infty} \tau^n(F)$. Then, $S = \overline{F} \cup (S - \overline{F}) \approx \tau(\overline{F}) \cup (S - \overline{F}) = S - F$.

This yields the sets $A = A_1 \cup A_2 \approx S - F \approx S$ and $B = B_1 \cup B_2 \approx S - F \approx S$, all by rotations, as we claimed in the beginning of the proof.

It is easy to extend the result from S to $P - \{\mathbf{0}\}$ by radial extension: append to each point $\mathbf{s}$ on the surface the segment $t\mathbf{s}$ with $0 < t \leqslant 1$. So there are subsets $\overline{A}$ and $\overline{B}$ of $P - \{\mathbf{0}\}$ such that $\overline{A} \cap \overline{B} = \emptyset$ and $\overline{A} \approx P - \{\mathbf{0}\} \approx \overline{B}$.

Finally, we need to extend the result from $P - \{\mathbf{0}\}$ to P. Let l be the line which passes through the point $(0, 0, r/2)$ on the z-axis and which is parallel to the x-axis. Call η the rotation about l by angle $\alpha = \arccos(\frac{1}{3})$. Then $H = \{\mathbf{0}, \eta(\mathbf{0}), \eta^2(\mathbf{0}), \eta^3(\mathbf{0}) \ldots\}$ are all distinct, because α is not commensurate with 2π. This implies that $P = H \cup (P - H) \approx \eta(H) \cup (P - H) = P - \{\mathbf{0}\}$, which finishes the proof. □

This, in and of itself, doesn't seem quite as counterintuitive; it is really the following two corollaries which illustrate how crazy the main result is.

Corollary 1. *Let P_0, P_1 and P_2 be three pairwise disjoint solid balls of the same radius r. Then $P_0 \approx P_1 \cup P_2$.*

Proof. By the Banach-Tarski Theorem, there are subsets $A, B \subseteq P_0$ such that $A \cap B = \emptyset$ and $A \approx P_0 \approx B$. Then $P_1 \cup P_2 \approx A \cup B \preccurlyeq P_0 \preccurlyeq P_1 \cup P_2$. By the above lemma, we have $P_0 \approx P_1 \cup P_2$. □

Corollary 2 (Banach-Tarski Paradox)**.** *Let $A, B \subseteq \mathbb{R}^3$ such that $P_1 \subseteq A \subseteq P_2$ and $P_3 \subseteq B \subseteq P_4$ for any solid balls P_1, P_2, P_3, P_4. In other words, let A and B be any two bounded subsets of $\mathbb{R}^3$ with nonempty interiors. Then $A \approx B$.*

Proof. Choose n large enough so that P_2 can be covered with n copies of P_3; call them $C_1, C_2, \ldots, C_n$. Put $C'_1 = P_2 \cap C_1$, $C'_2 = (P_2 \cap C_2) - C_1$, $C'_3 = (P_2 \cap C_3) - (C_1 \cup C_2), \ldots, C'_n = (P_2 \cap C_n) - (C_1 \cup C_2 \cup \cdots \cup C_{n-1})$. As a result of this construction, the sets $C'_1, \ldots, C'_n$ partition P_2. Translate them to disjoint copies of P_3; call them $D_1, D_2, \ldots, D_n$. Say C'_i moves to $D'_i \subseteq D_i$. Then $A \preccurlyeq P_2 \approx D'_1 \cup D'_2 \cup \cdots \cup D'_n \preccurlyeq D_1 \cup D_2 \cup \cdots \cup D_n \approx P_3$ by the above corollary, and $P_3 \subseteq B$, so that $A \preccurlyeq B$. By a repeat of the above process, we also find that $B \preccurlyeq A$. Thus, $A \approx B$ by the above lemma. □

It is worth noting that A and B, because they do not preserve volume, are nonmeasurable sets. This is a direct result of the use of the Axiom of Choice; the existence of nonmeasurable sets is actually nearly equivalent to the Axiom of Choice [2].

Conclusion

The Banach-Tarski Paradox is only one of many startling conclusions that follow from the most foundational, fundamental assumptions in mathematics. As mathematicians and mathematicians-in-training, one of the key things we learn is to subordinate our naïve intuition to rigorous logic; toward that end, familiarity with counterintuitive results is relatively important in continuing to develop mathematically. More importantly, though, they lend a surreal aesthetic to the structure of mathematics; they illustrate that our world is a lot wilder than we might have otherwise thought.

References

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