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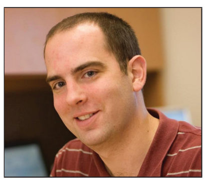
The Birank Number of $3 \times n$ Grid Graphs

Michael Fraboni and Shane Harder



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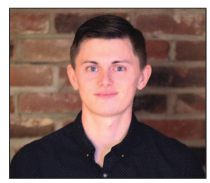

Abstract For any graph G, an assignment of ranks to its vertices by a function $f : V(G) \to \{1, 2, ..., k\}$ is a k-biranking of G if $f(u) = f(v)$ implies that every u-v path contains vertices x and y such that $f(x) > f(u)$ and $f(y) < f(u)$. The birank number of a graph, denoted $\mathrm{bi}(G)$, is the minimum k such that G has a k-biranking. We determine the birank number for $3 \times n$ grid graphs for several small values of n, and then obtain upper and lower bounds for all n. In the process we define two algorithms for constructing valid biranks on such graphs.

Introduction

A k-ranking on a graph G is an assignment of positive integers, or ranks, $1, \ldots, k$ to the vertices of G such that if any two vertices are assigned the same rank then every path between them contains a vertex with higher rank. Graph rankings were first used by Iyer, Ratliff, and Vijayan [7]. A k-biranking on a graph is a generalization of graph ranking which was defined by Jamison [8] as an assignment of ranks to the vertices of G so that if any two vertices are assigned the same rank then every path between them contains a vertex with higher rank and a vertex with lower rank. The minimum k for which a graph G has a valid k-biranking is the birank number of G which we denote by $\mathrm{bi}(G)$.

The birank number has been determined for path, cycle, ladder, and Möbius graphs ([5, 6]). This work focuses on $3 \times n$ grid graphs, denoted $G_{3,n}$. There has been some work on studying rankings on grid graphs ([1, 4, 9]), but these techniques do not generally extend to the birank problem.

A variation of ranking known as on-line ranking has been studied as well (see for instance [3]). An on-line ranking requires obtaining a valid ranking by assigning ranks to vertices as they are added in an arbitrary order to build a graph from its subgraphs. The ranking must be valid at each stage. We have not found any work discussing on-line biranking of graphs.

In the Preliminaries section we introduce some preliminary notions and determine the birank number for $G_{3,n}$ for $n \leq 6$. In the Growing Valid Birankings section, we introduce a method for constructing valid birankings on $G_{3,n}$ similar to the on-line rankings mentioned above in that it allows us to "grow" the grid by adding new columns but maintaining valid birankings. In the A Recursive Method section we illustrate a more efficient method for generating valid birankings on $G_{3,n}$ graphs. Finally, in the A Lower Bound section we establish a lower bound for $\mathrm{bi}(G_{3,n})$ and show that it differs from our upper bound by at most a factor of 3.

Throughout, we represent $G_{3,n}$ as a rectangular grid, three rows high, and will refer to rows and columns in this way.

Preliminaries

Here we introduce terminology useful in the proofs and discuss optimal birank numbers for $G_{3,1}$ through $G_{3,6}$ as base cases.

Dividers and Distances

The technique of high and low dividers has been used for other families of graphs such as paths and ladders (see [5] and [6]). The definition below is more general than those used in the above papers.

Definition 1. Given a biranking $f : V(G) \to \{1, ..., k\}$ on a graph G, and vertices x, y such that $f(x) = f(y)$, a *low divider* for x and y is a set of vertices $L \subseteq V(G)$ such that $\forall z \in L$, $f(z) < f(x)$ and the removal of L disconnects the graph with x and y in separate components.

A *high divider* is defined in a similar way using larger ranks. Clearly if x and y are assigned the same rank in a valid biranking, then there must be both a high and low divider between x and y.

For example, in Figure 1, the vertices assigned ranks 1 and 2 make up a low divider for the vertices assigned rank 3. Similarly the vertices assigned ranks 4 and 5 make up a high divider for the vertices assigned rank 3.

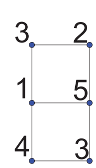

Figure 1: $G_{3,2}$

On $G_{3,n}$ a high or low divider for vertices x and y will consist of two or more vertices. If one of x or y is a corner vertex (i.e., a vertex of degree 2), then two vertices may suffice to form a divider. Otherwise a divider must contain at least three vertices. We will use the concept of dividers heavily in our later proofs.

We now note some properties involving the distance between vertices of equal rank. It is clear from the definition of a birank that for two vertices in any graph to be assigned the same rank they must be a distance of at least 3 apart.

Given a vertex v in a graph G, if we denote by N_v the set containing v together with the vertices adjacent to v, then the following lemma follows directly from the definition of birank:

Lemma 2. *Given a graph G with a valid biranking f, if $f(u) = f(v)$ for distinct vertices u and v in G, then $N_u \cap N_v = \varnothing$.*

For a biranking on a graph we denote by $n(r)$ the number of vertices assigned the rank r. The following lemma highlights the utility of a distance argument in establishing upper bounds for birank numbers.

Lemma 3. *Given any rank r in a valid biranking of $G_{3,5}$, $n(r) < 5$.*

Proof. Assume that $n(r) \geq 5$ on $G_{3,5}$. Notice that in $G_{3,n}$, if v is a corner vertex, then $|N_v| = 3$, otherwise $|N_v| \geq 4$. As a consequence of Lemma 2, it is clear that r may be assigned to at most two corner vertices. By Lemma 2, if $n(r) = 5$, then the total number of distinct vertices accounted for by the neighborhoods of vertices with rank r is at least $3 + 3 + 4 + 4 + 4 = 18$. However, $G_{3,5}$ has only 15 vertices and we have reached a contradiction. □

Base Cases

In general, we will find upper bounds for $\mathrm{bi}(G)$ by constructing a valid biranking on G, then we will prove this upper bound is tight. The birank number for $G_{3,2}$ through $G_{3,4}$ has been determined in [2]. Below, we determine the birank number for $G_{3,5}$ and $G_{3,6}$. See Table 1 for a summary of known birank numbers.

Theorem 4. $\mathrm{bi}\,(G_{3,5}) = 10$.

Proof. Assume that we have a valid biranking on $G_{3,5}$ with 9 ranks. Notice the ranks 1 and 2 may only appear at most once because there are not enough vertices assigned lower ranks to form a low divider. Similarly the ranks 8 and 9 may appear at most once. Now $n(3) \leq 2$ because there are at most two vertices assigned rank lower than 3, which means the rank 3 can have at most one low divider.

Note $n(4) \leq 2$ since if $n(3) = 2$ then a vertex of rank 3 must be placed in a corner with 1 and 2 directly adjacent, so there are no ranks lower than 4 available to form a low divider and $n(4) = 1$. If $n(3) = 1$, then there are only three vertices with ranks lower than 4, so the rank 4 has at most one low divider.

Analogously, $n(7) \leq 2$ and $n(6) \leq 2$ by a high divider argument. Given these restrictions, the ranks 1 through 4 and 6 through 9 have accounted for at most 12 vertices, so we need at least three vertices assigned a rank of 5. We will show this is impossible.

Case 1. $n(3) = n(4) = 2$ or $n(6) = n(7) = 2$:

If $n(3) = 2$, then a vertex of rank 3 must be placed in a corner with 1 and 2 directly adjacent because we need to form a low divider using only two vertices.

Now in order to have $n(4) > 1$ the rank 4 needs to have a low divider. The only way this can happen is if 4 is placed in the corner nearest the existing rank 3 and the second 3 is placed directly adjacent to 4. See Figure 2.

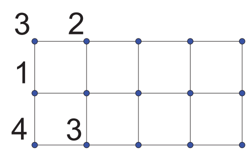

Figure 2: $G_{3,5}$

Now we have a second 4 to use as part of a low divider for 5, but the only way $n(5) > 1$ is if the 5 is placed to the right of 1 with a second 4 placed directly to the right of it. See Figure 3.

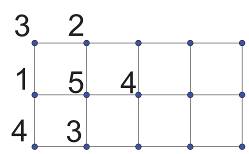

Figure 3: $G_{3,5}$

Notice, 5 is surrounded by lower ranks and free to repeat. However, there are no more ranks less than 5 to allow another repeat. Therefore, $n(5) \leq 2$ if $n(3) = n(4) = 2$. Likewise if $n(7) = n(6) = 2$, then $n(5) \leq 2$ by a similar argument using high dividers. Therefore, ranks 1 through 9 can be used at most 14 times on 15 vertices and we have reached a contradiction.

Case 2. $n(3) + n(4) \leq 3$ and $n(6) + n(7) \leq 3$:

If $n(3)+n(4) \leq 3$ and $n(6)+n(7) \leq 3$, then ranks 1 through 4 and 6 through 9 account for at most 10 vertices. On $G_{3,5}$, $n(5) \leq 4$ by Lemma 3. Therefore, ranks 1 through 9 can be used at most 14 times on 15 vertices and we have reached a contradiction.

Therefore, $\text{bi}(G_{3,5}) > 9$. By Figure 4 we see $\text{bi}(G_{3,5}) \leq 10$. Therefore, $\text{bi}(G_{3,5}) = 10$. □

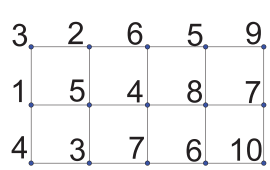

Figure 4: Optimal biranking on $G_{3,5}$

Theorem 5. $\text{bi}(G_{3,6}) = 11$.

Proof. Assume that we have a biranking on $G_{3,6}$ with 10 ranks. Notice that the ranks 1, 2, 9, 10 may appear at most once. Note that $n(3) \leq 2$. Similarly $n(8) \leq 2$.

Case 1. $\boldsymbol{n(3) = n(8) = 2}$**:**

Notice that if $n(3) = 2$, then a 3 must be placed in a corner and the adjacent vertices to the rank 3 must be assigned the ranks 1 and 2. This will restrict $n(4) \leq 2$ and $n(5) \leq 2$ by an argument similar to the one given in the proof of Theorem 4. Similarly, if $n(8) = 2$, then $n(7) \leq 2$ and $n(6) \leq 2$. Notice this will only account for at most 16 out of the 18 vertices in $G_{3,6}$. Thus we have a contradiction.

Case 2. $\boldsymbol{n(3) = 2}$ **and** $\boldsymbol{n(8) \leq 1}$ **or** $\boldsymbol{n(3) \leq 1}$ **and** $\boldsymbol{n(8) = 2}$**:**

Notice that if $n(3) = 2$, then $n(4) \leq 2$, $n(5) \leq 2$ by the same argument as in Case 1. However, if $n(8) \leq 1$, then $n(7) \leq 2$ and thus $n(6) \leq 3$ because there are at most 4 vertices with ranks larger than 6 allowing for at most two high dividers. Notice that this will only account for at most 16 out of the 18 vertices in $G_{3,6}$. If $n(3) \leq 1$ and $n(8) = 2$, the same result will follow. Thus we have a contradiction.

Case 3. $\boldsymbol{n(3) \leq 1}$ **and** $\boldsymbol{n(8) \leq 1}$**:**

Notice that if $n(3) \leq 1$, then $n(4) \leq 2$ because there are at most 3 lower ranks than 4. Now, $n(5) \leq 3$ because there are at most 5 lower ranks allowing for at most 2 low dividers. Similarly, by higher ranks, if $n(8) \leq 1$, then $n(7) \leq 2$ and $n(6) \leq 3$. Notice that this only accounts for 16 out of the 18 vertices in $G_{3,6}$. Thus, we have reached a contradiction.

Therefore, $\operatorname{bi}(G_{3,6}) > 10$. By Figure 5, we see that $\operatorname{bi}(G_{3,6}) \leq 11$. Therefore $\operatorname{bi}(G_{3,6}) = 11$. □

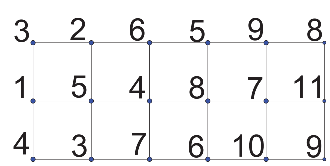

Figure 5: Optimal biranking on $G_{3,6}$

Now we have a set of base cases where the birank number of $G_{3,n}$ has been determined for $n < 7$.

We finish what we will consider our base cases by providing figures below which establish the bounds $\operatorname{bi}(G_{3,7}) \leq 12$ and $\operatorname{bi}(G_{3,8}) \leq 13$.

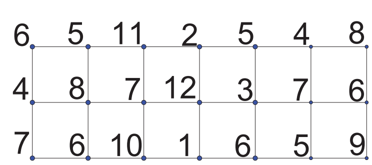

Figure 6: Biranking on $G_{3,7}$

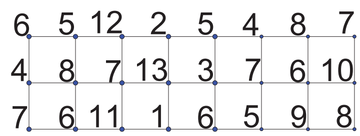

Figure 7: Biranking on $G_{3,8}$

Growing Valid Birankings

We now develop a method for building valid birankings on $G_{3,n}$ in general, which is also optimal from $G_{3,2}$ to $G_{3,6}$, called the "3 in the corner" or 3C Method. This is a constructive method that begins with a small base case and describes how to assign ranks as the graph adds columns.

We begin with $G_{3,2}$, as labeled in Figure 8, as a base case.

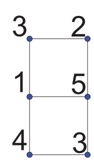

Figure 8: $G_{3,2}$ labeled using the 3C Method.

Given $G_{3,n}$ for $n > 2$ with a biranking constructed using the 3C method, we may extend this ranking to $G_{3,n+1}$ using the two cases below.

Case 1. n is even:

Given a biranking on $G_{3,n}$ constructed by the 3C Method with t as the largest label, when n is even, the $(n-1)$ and n columns will be assigned ranks as in Figure 9.

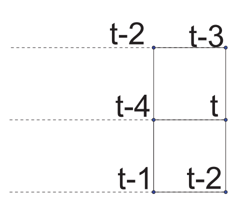

Figure 9: A labeling of the $(n-1)$ and n columns of $G_{3,n}$ by the 3C Method for n even.

We will extend this labeling to $G_{3,n+1}$ as in Figure 10.

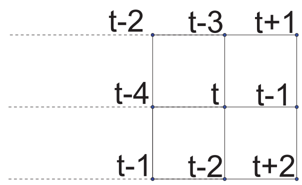

Figure 10: A labeling of the $(n-1)$, n, and $(n+1)$ columns of $G_{3,n+1}$ by the 3C Method for n even.

Note that by assumption, the ranks $(t+1)$ and $(t+2)$ do not appear earlier in the graph so they may appear anywhere in the $(n+1)$ column. Notice that $(t-1)$ has high and low dividers and so may appear on the middle vertex in the $(n+1)$ column. Thus our extension is a valid biranking on $G_{3,n+1}$.

Case 2. n is odd:

Let a biranking on $G_{3,n}$ be constructed by the 3C Method with t as the largest label. When n is odd, the $(n-1)$ and n column will be assigned as in Figure 11.

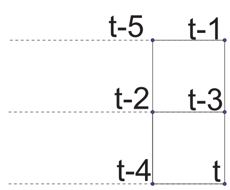

Figure 11: A labeling of the $(n-1)$ and n columns of $G_{3,n}$ by the 3C Method for n odd.

We will extend this labeling to $G_{3,n+1}$ as in Figure 12.

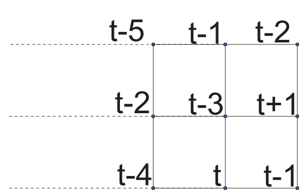

Figure 12: A labeling of the $(n-1)$, n, and $(n+1)$ columns of $G_{3,n+1}$ by the 3C Method for n odd.

Note that the rank $(t+1)$ may appear on any vertex in the $(n+1)$ column. All other repeated ranks have appropriate high and low dividers. Thus our extension is a valid biranking on $G_{3,n+1}$.

Definition 6. Let $T(G_{3,n})$ denote the number of ranks used to birank $G_{3,n}$ using the 3C Method.

The following lemma is clear from the definition of the 3C Method.

Lemma 7. *Given $G_{3,n}$ labeled using the 3C Method,*

$$T(G_{3,n}) = T(G_{3,n-1}) + \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

This recursive formula gives a sequence of values for $T(G_{3,n})$ which we may express as follows.

Theorem 8. $T(G_{3,n}) = 5 + \left\lfloor \frac{n-2}{2} \right\rfloor + 2\left\lceil \frac{n-2}{2} \right\rceil$.

Proof. Let $G_{3,2}$ serve as a base case where $T(G_{3,2}) = 5$. Notice that for $n > 2$, a 1 will be added to $T(G_{3,2})$ for every even number between 2 and n. Similarly a 2 will be added to $T(G_{3,2})$ for every odd number between 2 and n. The number of evens larger than 2 and less than or equal to n is given by $\left\lfloor \frac{n-2}{2} \right\rfloor$. Likewise the number of odds larger than 2 and less than or equal to n is given by $\left\lceil \frac{n-2}{2} \right\rceil$. Therefore $T(G_{3,n}) = 5 + \left\lfloor \frac{n-2}{2} \right\rfloor + 2\left\lceil \frac{n-2}{2} \right\rceil$. □

Clearly, $\mathrm{bi}(G_{3,n}) \leq T(G_{3,n})$. The 3C Method does very well for small values of n and in fact it generates optimal birankings on $G_{3,2}$ through $G_{3,6}$. However it stops being optimal at $G_{3,7}$ since every rank in a biranking generated by the 3C Method may appear at most twice. Other methods for constructing birankings allow for individual ranks to repeat more often which becomes important for larger values of n.

A Recursive Method

We now develop a recursive method for generating valid birankings on $G_{3,n}$. The Straight Cut Method uses a central cut to divide the graph into two smaller subgraphs. First we will introduce the definition of the Straight Cut and show how it acts on $G_{3,n}$ in general, then we compute the number of ranks needed to birank $G_{3,n}$ using the Straight Cut Method.

Definition 9. The *Straight Cut* is a subgraph of $G_{3,n}$ which appears in the $\left\lceil \frac{n}{2} \right\rceil$ and $\left\lceil \frac{n}{2} + 1 \right\rceil$ columns as labeled in figure 13.

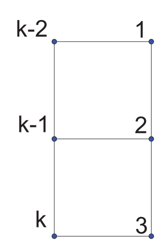

Figure 13: The Straight Cut

For large enough k, the Straight Cut acts as high and low dividers and splits $G_{3,n}$ leaving two subgraphs A and B as in figure 14. Thus A and B may be approached as two independent biranking problems (as long as no ranks smaller than 4 or larger than $k - 3$ are used).

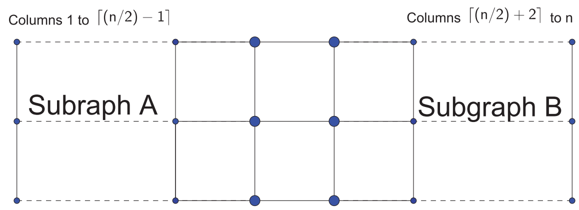

Figure 14: The Straight Cut with Subgraphs A and B

Since the number of columns in subgraph B is at most equal to the columns in A, it suffices to find a valid biranking for A, then apply the same pattern of ranks to B.

More precisely, let $f : V\left(G_{3,\lceil \frac{n}{2}-1\rceil}\right) \to \{1, ..., s\}$ be a valid biranking on A. Then subgraph B may also be labeled with at most s ranks. If we let $k = s + 6$, then $g : V(G_{3,n}) \to \{1, ..., (s + 6)\}$ is a valid biranking if $g(v) = f(v) + 3$ for subgraphs A and B on $G_{3,n}$ and if g assigns the values of Figure 13 to the vertices of the straight cut.

Therefore, if $G_{3,\lceil \frac{n}{2}-1\rceil}$ may be labeled with s ranks, then $G_{3,n}$ may be labeled with $s + 6$ ranks. We have proven the following lemma.

Lemma 10. $\mathrm{bi}(G_{3,n}) \le \mathrm{bi}\left(G_{3,\lceil \frac{n}{2}-1\rceil}\right) + 6.$

The Straight Cut method applies this cutting process to $G_{3,n}$, then again to the larger subgraph, then again until we reach one of our base cases. After t iterations of this method on $G_{3,n}$, we are left with $G_{3,b}$, where

$$\begin{aligned}
b &= \left\lceil \frac{1}{2}\left\lceil \cdots \left\lceil \frac{1}{2}\left\lceil \frac{1}{2}\left\lceil \frac{n}{2} - 1\right\rceil - 1\right\rceil - 1\right\rceil \cdots \right\rceil - 1\right\rceil \\
&= \left\lceil \frac{1}{2}\left(\cdots \left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{n}{2} - 1\right) - 1\right) - 1\right) \cdots \right) - 1\right\rceil \\
&= \left\lceil \frac{1}{2^t}\left(n - \sum_{i=1}^{t-1} 2^i\right) - 1\right\rceil \\
&= \left\lceil \frac{1}{2^t}\left(n + 1 - \sum_{i=0}^{t} 2^i\right)\right\rceil \\
&= \left\lceil \frac{1}{2^t}\left(n + 1 + \left(1 - 2^{t+1}\right)\right)\right\rceil \\
&= \left\lceil \frac{n+2}{2^t}\right\rceil - 2.
\end{aligned}$$

If the set of base cases for $\mathrm{bi}(G_{3,n})$ is $G_{3,1}$,...,$G_{3,m}$, then the number of steps this method needs to reach a base case $G_{3,b}$ such that $b \le m$ is given by

$$\left\lceil \frac{n+2}{2^t} \right\rceil - 2 \le m, \text{ which yields}$$
$$\left\lceil \frac{n+2}{m+2} \right\rceil \le 2^t, \text{ and so}$$
$$\left\lceil \log_2 \left(\frac{n+2}{m+2} \right) \right\rceil \le t.$$

For the Straight Cut method we use the set of base cases from the Preliminaries section, and so $m = 8$. Thus the number of iterations to reach a base case is given by $t = \left\lceil \log_2 \left(\frac{n+2}{10} \right) \right\rceil$. After t iterative cuts on $G_{3,n}$, if $G_{3,(\lceil \frac{n+2}{2^t} \rceil - 2)}$ can be labeled with s ranks, then $G_{3,n}$ may be labeled with $s + 6t$ ranks.

Definition 11. Given a graph $G_{3,n}$, we denote by $S(G_{3,n})$ the number of ranks needed to produce a valid biranking using the Straight Cut method.

Theorem 12. *Given $n \ge 5$, if $b = \lceil \frac{n+2}{2^t} \rceil - 2$ and $t = \left\lceil \log_2 \left(\frac{n+2}{10} \right) \right\rceil$, then*

$$S(G_{3,n}) = 6t + \begin{cases} 8 & \textit{if } b = 4 \\ 10 & \textit{if } b = 5 \\ 11 & \textit{if } b = 6 \\ 12 & \textit{if } b = 7 \\ 13 & \textit{if } b = 8. \end{cases}$$

Proof. As discussed above, if we iterate the Straight Cut process t times on $G_{3,n}$, we are left with the base case of $G_{3,b}$ with $4 \le b \le 8$. The cases in the formula above indicate how many ranks are needed for each base case. Each time we iterate the process we require 6 more ranks which will add $6t$ to the total. □

Clearly $\mathrm{bi}(G_{3,n}) \le S(G_{3,n})$ and so this function gives an upper bound for all $3 \times n$ grid graphs and shows the $\mathrm{bi}(G_{3,n})$ grows with at most the log of n.

A Lower Bound

The previous sections have focused on developing methods for generating valid birankings on $G_{3,n}$ which provide an upper bound for $\mathrm{bi}(G_{3,n})$. Here we establish a lower bound on $\mathrm{bi}(G_{3,n})$.

First, note that for n even, the cycle graph C_{3n} is a subgraph of $G_{3,n}$, and for $n \ge 3$ odd, $G_{3,n}$ has C_{3n-1} as a subgraph. See Figure 15 for examples.

The birank number of a cycle graph is known (see [5]), so we have the bounds given below.

Lemma 13. *When n is even,*

$$\mathrm{bi}(G_{3,n}) \ge \mathrm{bi}(C_{3n}) = \lfloor \log_2(3n-1) \rfloor + \left\lfloor \log_2 \left(\frac{3n-1}{3} \right) \right\rfloor + 3.$$

When n is odd,

$$\mathrm{bi}(G_{3,n}) \ge \mathrm{bi}(C_{3n-1}) = \lfloor \log_2(3n-2) \rfloor + \left\lfloor \log_2 \left(\frac{3n-2}{3} \right) \right\rfloor + 3.$$

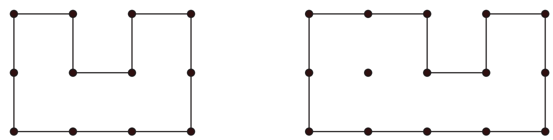

Figure 15: Examples showing how we may draw C_{12} as a subgraph of $G_{3,4}$ and C_{14} as a subgraph of $G_{3,5}$.

Values of this lower bound for small n are shown in Table 1. We have the following result that shows this lower bound comes within a constant factor of the upper bound given by the Straight Cut method.

Theorem 14. *If* $\mathrm{lb}(n)$ *is the lower bound for* $\mathrm{bi}(G_{3,n})$ *given above, then* $S(G_{3,n}) \leq 3 \cdot \mathrm{lb}(n)$ *for all* $n \geq 5$.

Proof. First, Table 1 establishes the result for $n = 5$. Now note that

$$\begin{aligned} \lceil \log_2(n+2) \rceil &\leq \lfloor \log_2(n+2) + 1 \rfloor \\ &= \lfloor \log_2(2n+4) \rfloor \\ &= \lfloor \log_2(2n+6-2) \rfloor \\ &\leq \lfloor \log_2(3n-2) \rfloor, \end{aligned}$$

where the last inequality holds when $n \geq 6$. So for $n \geq 6$,

$$\begin{aligned} \frac{S(G_{3,n})}{\mathrm{lb}(n)} &\leq \frac{6\left\lceil \log_2\left(\frac{n+2}{10}\right)\right\rceil + 13}{\lfloor \log_2(3n-2) \rfloor + \left\lfloor \log_2\left(\frac{3n-2}{3}\right)\right\rfloor + 3} \\ &\leq \frac{6\left\lceil \log_2\left(\frac{n+2}{8}\right)\right\rceil + 13}{\lfloor \log_2(3n-2) \rfloor + \left\lfloor \log_2\left(\frac{3n-2}{4}\right)\right\rfloor + 3} \\ &\leq \frac{6\lceil \log_2(n+2) \rceil - 5}{\lfloor \log_2(3n-2) \rfloor + \lfloor \log_2(3n-2) \rfloor + 1} \\ &\leq \frac{6\lfloor \log_2(3n-2) \rfloor - 5}{2\lfloor \log_2(3n-2) \rfloor + 1} \\ &< \frac{6\lfloor \log_2(3n-2) \rfloor}{2\lfloor \log_2(3n-2) \rfloor} \\ &= 3. \end{aligned}$$

□

Summary

Table 1 summarizes the number of ranks needed to produce a valid biranking on $G_{3,n}$ using each of our methods. Numbers that are proven to be the optimal birank number are in bold. Numbers which represent our best known upper bound for $\mathrm{bi}(G_{3,n})$ are indicated with an asterisk.

Note that the 3C method does well in the beginning, but is ultimately inefficient for larger graphs since it will never use a rank more than twice. The Straight Cut method gives us our best upper bounds at the moment, but the 3C method is still interesting in that it provides a valid ranking which grows with the graph. For example, the assignment of ranks to $G_{3,12}$ and $G_{3,13}$ are significantly different when using the Straight cut method, while adding the extra row has no impact on most of the graph using the 3C method.

n	Base Cases	3C method	Straight Cut method	Lower bound
2	**5**	**5**	-	5
3	**7**	**7**	-	6
4	**8**	**8**	-	7
5	**10**	**10**	**10**	8
6	**11**	**11**	**11**	9
7	*12**	13	*12**	9
8	*13**	14	*13**	9
9	-	16	*14**	10
10	-	17	*14**	10
11	-	19	*16**	11
12	-	20	*16**	11
13	-	22	*17**	11
14	-	23	*17**	11
15	-	25	*18**	11

Table 1: Number of ranks needed to produce a valid biranking on $G_{3,n}$ with each method. Bold numbers are proven to be optimal. An asterisk indicates our best known upper bound for $\mathrm{bi}(G_{3,n})$.

Future work in this area might include either proving the optimality of the Straight Cut method, or developing similar algorithms which cut the graph into smaller components in different ways – perhaps with diagonal cuts instead of vertical ones. Potentially, a more efficient algorithm might be found. Additionally, finding a tighter lower bound would be of interest.

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Bibliography

[1] Hannah Alpert, *Rank numbers of grid graphs*, Discrete Math. **310** (2010), no. 23, 3324–3333.

[2] Andrew Benson, *Birank number of operationally constructed graphs*, 2013, Moravian College, Honors Thesis.

[3] Erik Bruoth and Mirko Horňák, *On-line ranking number for cycles and paths*, Discuss. Math. Graph Theory **19** (1999), no. 2, 175–197, The Seventh Workshop "3in1" Graphs '98 (Krynica).

[4] Sitan Chen, *On the rank number of grid graphs*, Preprint, arXiv:1208.1814 (2012).

[5] Michael J. Fisher, Nickolas Fisher, Michael Fraboni, and Darren A. Narayan, *The birank number of a graph*, Congr. Numer. **204** (2010), 173–180.

[6] Michael Fraboni and Madison Zebrine, *The birank number of ladder, prism and möbius ladder graphs*, Australas. J. Combin. **63** (2015), no. 1, 34–40.

[7] Ananth V. Iyer, H. Donald Ratliff, and G. Vijayan, *Optimal node ranking of trees*, Inform. Process. Lett. **28** (1988), no. 5, 225–229.

[8] Robert E. Jamison, *Coloring parameters associated with rankings of graphs*, Congr. Numer. **164** (2003), 111–127.

[9] Sarah Novotny, Juan Ortiz, and Darren A. Narayan, *Minimal k-rankings and the rank number of P_n^2*, Inform. Process. Lett. **109** (2009), no. 3, 193–198.