Student-Faculty Seminar

The Distribution of Primes

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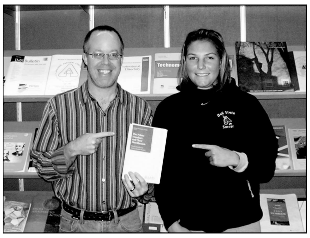
To understand the natural numbers, we must first understand the primes. Yet the behavior of the primes remains for the most part a mystery. For example, despite the efforts of many great mathematicians over hundreds of years, we still know very little about how the primes are *distributed* (scattered) throughout the natural numbers.



The results and conjectures pertaining to the distribution of the primes, known as *prime distribution theory*, was the topic chosen for the fall 2004 student-faculty seminar. With such a challenging topic, the group experienced both the heights of inspiration and the depths of confusion. What follows is a description of various aspects of the seminar, including both highlights and 'lowlights.'

Highlight: connections

One way to get a feeling for where the primes are within the natural numbers is to count the number of primes $\pi(n)$ less than or equal to a given number n. There is no real hope for finding a simple rule for $\pi(n)$, but the *Prime Number Theorem*[1] (PNT) says that $\pi(n)$ can be approximated by the logarithmic integral function

$$\mathrm{Li}(n) := \int_2^n \frac{1}{\log t}\, dt.$$

More specifically, PNT says that

$$\lim_{n\to\infty} \frac{\mathrm{Li}(n)}{\pi(n)} = 1. \tag{1}$$

[1] The Prime Number Theorem was first proved in 1896 by Hadamard and Valleé-Poussin (see [1]). Fifty years earlier, Chebyshev [2] proved that if the limit in PNT exists, its value must be one. Fifty years later, Erdös [4] and Selberg [9] gave 'elementary' proofs of PNT that bypassed the zeta function.

The Prime Number Theorem is the hallmark theorem of prime distribution theory, and much of the seminar was devoted to investigating why PNT is true. Our path to PNT highlighted a beautiful and powerful connection with analysis. The starting point is the fact (due to the Fundamental Theorem of Arithmetic and known to Euler) that if $s = \sigma + it$ is a complex number and $\sigma > 1$ then

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{2}$$

where the product (called an *Euler product*[2]) is taken over all prime numbers p. Equation (2) defines a complex differentiable function on the half-plane $\sigma > 1$. Through the miracle of complex analysis, this function can be extended uniquely to a function $\zeta(s)$ (called the Riemann zeta function) which is complex differentiable[3] for $s \neq 1$. An intimate connection emerged between the zeros of $\zeta(s)$ and the Prime Number Theorem. In particular, we discovered that the Prime Number Theorem is a consequence of the fact that $\zeta(s)$ has no zeros on the line $\sigma = 1$. (The first proofs of PNT by Hadamard and Valeé-Poussin employed this method.)

We saw that there are further fascinating and surprising applications of the zeta function. Let p_n be the n-th prime. If the logarithmic integral $\mathrm{Li}(x)$ did a perfect job of counting primes, then $\mathrm{Li}(p_n)$ would be n on the nose. However, $\mathrm{Li}(x)$ is merely an approximation, and the *Riemann Hypothesis* (first formulated by Riemann in 1859) is principally a conjecture about the growth of the associated error term $\mathrm{Li}(p_n) - n$. The Riemann Hypothesis[4] asserts

$$\mathrm{Li}(p_n) = n + O(n^{1/2+\epsilon}) \qquad \text{for every } \epsilon > 0. \tag{3}$$

Just as in the proof of PNT, there is a connection between the Riemann Hypothesis (3) and the zeros of the zeta function $\zeta(s)$. Specifically, except for zeros at $s = -2, -4, -6, \ldots$, all other zeros of $\zeta(s)$ must lie in the 'critical strip': $\{s = \sigma + it \mid 0 < \sigma < 1\}$. If it is further true that if all of these critical zeros lie on the line $\sigma = 1/2$, then the Riemann Hypothesis (3) will hold, and vice versa[5].

Lowlight: difficulty and details

The book [5] we used for the seminar was a small, well-written paperback measuring approximately seven millimeters in thickness. This book was both

[2] We used the Euler product in the earliest stages of the seminar to show that $\sum \frac{1}{p}$ diverges. From a probabilistic standpoint, if we set $s = 1$ then the reciprocal partial Euler products over $p < \sqrt{n}$ can be viewed as the probability that n is prime.

[3] Since $\sum \frac{1}{n}$ diverges, we can deduce from (2) that $\zeta(s)$ is badly behaved at $s = 1$, but the behavior isn't too awful: one can show that $(s-1)\zeta(s) \to 1$ as $s \to \infty$. We say that $\zeta(s)$ has a *simple pole* at $s = 1$.

[4] Regarding the 'O' notation in the Riemann Hypothesis: let f, g be real-valued functions on $\mathbb{R}$. We say $f = O(g)$ if there is a positive constant C satisfying $|f(x)| \leqslant C|g(x)|$ for all $x \in \mathbb{R}$.

[5] Showing that all the critical zeros of the zeta function lie on the line $\sigma = 1/2$ is a holy grail for number theorists. The Clay Mathematics Institute is currently offering a million dollars for a correct proof of the Riemann Hypothesis.

good and bad for us: good, in that it contained only important, overarching ideas (including lucid, motivating descriptions), and bad, in that many hard details were swept under the rug. Since we had no resident prime distribution specialist, the lack of details and relative difficulty of the material often plunged us into darkness and confusion. Sometimes, however, even the grinding details could be inspiring. For example, when interchanging the order of two Riemann-Stieltjes integrals as a part of the proof of Mertens' Theorem[6], the faculty could primly point out the value of Maths 472!

Finale: probabilistic models

Our difficulties with prime distribution theory served as a perfect backdrop to a latter portion of the seminar, which was devoted to probabilistic models for the primes. Models, which ideally provide an approximation of reality which is simple enough to understand, often come to our aid when the original phenomenon is overly complex.

In the 1930's, Harald Cramér [3] introduced a simple (and today widely known) model for the primes in which a natural number n is declared to be 'prime' with probability $\frac{1}{\log n}$, and the 'primality' of n is independent of the 'primality' of previous numbers[7]. Cramér's model is featured in the famous heuristic argument in favor of the Twin Primes Conjecture[8]: by independence, a pair $n, n+2$ of natural numbers are both prime with probability $[\log n \cdot \log(n+2)]^{-1}$, so the number of pairs of twins less than n is approximately

$$\int_2^n \frac{1}{\log t \cdot \log(t+2)}\, dt. \tag{4}$$

Since the integral (4) tends to infinity as $n \to \infty$, this argues in favor of the existence of infinitely many twin prime pairs[9].

In counterpoint to the Cramér model, David Hawkins [7] introduced an elegant probabilistic model (not discussed in the seminar) based on a randomized version of the sieve of Eratosthenes. Over the past fifty years, the Hawkins model has been used to predict the truth, in the strongest probabilistic sense, of results (both established and conjectured) concerning the distribution of the prime numbers, including the Twin Primes Conjecture and the Riemann Hypothesis (e.g., see [8]).

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[6] As I'm sure you recall from the seminar, Mertens' Theorem states that
$\prod_{p \leqslant x} (1 - 1/p)^{-1} \sim e^{\gamma} \log x$, where γ is Euler's constant and $\sim$ is asymptotic equivalence.

[7] The Prime Number Theorem is the basis for this modeling assumption: PNT asserts that $\int_2^n \frac{1}{\log t}\, dt$ does a reasonably good job of counting the primes less than n, so, viewing $\frac{1}{\log t}$ as an approximate density function for the primes, we conclude that $\frac{1}{\log n}$ is more or less the probability that n is prime.

[8] Successive primes p_n and p_{n+1} are *twins* if $p_{n+1} - p_n = 2$. The Twin Primes Conjecture asserts that there are infinitely many pairs of twin primes.

[9] Integration by parts in (4) suggests that the number of twin prime pairs less than n is asymptotically equivalent to $\frac{n}{log^2 n}$. This essentially matches the conjecture made by Hardy and Littlewood [6] without appealing to probability.

References

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