The Efficacy of a Frequency-Interval Model Applied to Byzantine Music

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Abstract The songs found in Byzantine music possess great cultural significance, a beautiful sound, and wonderful mathematics. The purpose of this paper is to show that Byzantine music satisfies a similar exponential relation $F = \frac{c}{(I+1)^D}$ that some classical music pieces studied by K. Hsü and A. Hsü in [2] satisfy. Here F denotes the frequency of a note interval I between successive notes, D is the dimension of the model, and c is a proportionality constant.

Introduction

In a 1990 paper [2], K. Hsü and A. Hsü studied three classical music pieces namely Bach's Toccata, Mozart's Sonata, and six Swiss children's songs. For each piece they recorded the percentage of frequency (F) of a certain note interval I between successive notes. That is, they counted how often is a note one, two, three, ... semitone(s) above or below its previous note. They referred to those numbers as the frequency F of the incidents $I = 1, 2, 3, \ldots$ respectively. They showed that each piece satisfies a "fractal" relation of the form $F = \frac{c}{I^D}$, where D is a "fractal" dimension and c is a constant of proportionality.

In this paper we follow the footsteps of K. Hsü and A. Hsü and show that Byzantine music satisfies a similar exponential model $F = \frac{c}{(I+1)^D}$. The horizontal shift by one unit allows us to avoid taking the logarithm of zero when constructing a log-log plot.

Byzantine Music Historical Background

Byzantine music is an ancient form of writing music with roots stemming all the way back to Pythagorus. The nature of the music gives it a unique monophonic sound. Today, the music's main purpose is for worship in the Eastern Orthodox Church, though in the past it had some entertainment purposes outside of the church. Byzantine music began to grow in 330 CE, the same year that the Emperor Constantine enforced the practice of toleration for Christianity in the Roman Empire. In its earliest forms, Byzantine music was influenced by the music of the age, including Jewish and Greek music. The Greek heritage of music provided the mechanism for dividing the scales, using the technique which stemmed all the way back to Pythagorus himself. In the Jewish synagogue, the Psalms were chanted using only the human voice. This is where the monophonic sound of Byzantine music originated from [11]. Through songs, the biblical stories could be presented to everyone, including those who could not read and putting the stories to a melody made the stories even easier to recall [11]. The music was passed along vocally from generation to generation without being written down. As the church began to spread, the singers' memories were no longer trusted to accurately recall the chants. So, the music was written down around the 9th century. The written form of Byzantine music has evolved over the years, but the form that is used today developed around the last quarter of the 12^{th} century [1].

Reading the Music

There are eight different Tones to Byzantine music, each with a different feeling or emotion and a different musical sound. The eight different Tones arose from the eight ecclesiastical modes that were already present in the 8th century [1]. Each Tone has certain rules governing the use of sharps and flats.

Byzantine music notation is quite different from the five line staff Western notation. There are different rules and symbols used in Byzantine music. At the beginning of each piece, there is a symbol which tells the chanter which Tone number to use and which note to start with, and yet another symbol for the tempo of the piece. Instead of telling the chanter what note to sing next, Byzantine music uses symbols to tell the chanter how many steps to move up or down along the music scale from the previous note, i.e., an iterative definition.

The Diatonic scale used in Byzantine music is very similar to the one used in Western music [20]. It begins with $N\eta$, which corresponds to the note C (see Figure 1).

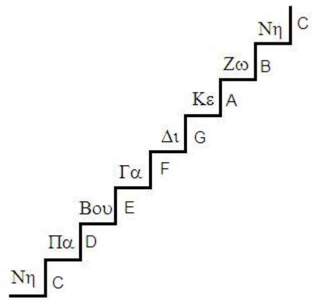


Figure 1: The Byzantine music scale. The notes on the left are the notes from Byzantine music and the notes on the right are the corresponding notes in Western music.

The Western Chromatic and Byzantine Diatonic scales are equal-tempered (i.e., the notes are all equally spaced), with the ratio of frequencies of consecutive notes being $\sqrt[12]{2}$. To switch to the Soft Chromatic scale, Byzantine music uses certain symbols to redefine the frequencies of notes on the scale. For example in the Soft Chromatic scale Pa ($\Pi\alpha$) and Ga ($\Gamma\alpha$) are executed with a sharp.

To get a feeling for how Byzantine music works, we will demonstrate the process using a small section of a song in Tone 1, shown below.

Figure 2: A line of a song in Tone 1.

The symbol at the beginning tells us that we should start at the note $\Pi\alpha$, which corresponds to the note D. The next symbol is called an Ison ($I\sigma o\nu$) and tells us that next note does not move up or down. So the first note is $\Pi\alpha$ or D. The next symbol is a combination of a Oligon ($O\lambda\acute{\iota}\gamma o\nu$) with a Kentima ($K\acute{\varepsilon}\nu\tau\eta\mu\alpha$) above it. The Oligon moves the note up 1 step on the scale and the Kentima moves it 2 steps up. Thus, we move up on the scale 3 steps to the note $\Delta\iota$ or G. Next, we have two Isons in a row, which means that the next two notes would also be $\Delta\iota$ or G. The next symbol is called Apostrophos ($A\pi\acute{o}\sigma\tau\rho o\varphi o\varsigma$), and it moves the note one step down to $\Gamma\alpha$ or F. From there, the Oligon moves us up one step to $\Delta\iota$ or G. The next symbol is called Petasti ($\Pi\iota\tau\alpha\sigma\tau\acute{\eta}$), and it moves us up one step (with strength and flickering sound) to $K\varepsilon$, or A. The beautiful flickering characteristic of the Petasti gives a unique sound to Byzantine music. For the purposes of this paper, we ignore the flickering action of the Petasti and treat it as an Oligon. Summarizing the notes gives us $\Pi\alpha$, $\Delta\iota$, $\Delta\iota$, $\Delta\iota$, $\Gamma\alpha$, $\Delta\iota$, $K\varepsilon$, $\Delta\iota$, $\Delta\iota$, $\Gamma\alpha$, $\Delta\iota$, $K\varepsilon$. Anyone with a knowledge of both Byzantine music and Western music can translate between the two forms. Now that we have a basic grasp on how the music works, we can begin analyzing the music. For a more complete resource of the rules in Byzantine music see [21].

Data Collection

For the purpose of our study, we chose eight Byzantine music pieces, one from each of the eight Tones. We then followed the footsteps of K. Hsü and A. Hsü in [2]. That is, we started by decoding the pieces as described in the previous section. This created a sequence of notes which corresponded to a sequence of semitone frequencies s_n (see Table 2 in Appendix). From the sequence of semitone frequencies s_n, we define the sequence of interval length between consecutive frequencies I_n by

$$I_n = |s_{n-1} - s_n|.$$

For the chosen pieces, we noticed that for all n, $0 \leq I_n \leq 12$. In general $0 \leq I_n \leq M$ for some maximum interval length M. It is expected that M is a small positive integer as Byzantine music is designed to be chanted by a human voice.

Next, we count how often the interval length I $(0 \leq I \leq M)$ occurred within the music piece and denote that number by f_I.In the cases of a nonzero f_I, we define the percentage frequency of the interval length I by

$$F_I = \frac{f_j}{\sum f_j} \cdot 100$$

We ignore intervals with zero frequency. The data collected from a piece in Tone 7, "Inna Hirodos" (see Appendix) can be found in [4] and Table 3 in Appendix . One can listen to this music piece on Youtube at [12].

For each piece, we entered the semitone frequency sequence s_n in the JFugue package for Java™[3]. JFugue produced a music file that we compared closely with the actual chanted music and checked that our decoded data played the same music, and hence our collected data was verified.

Analysis

K. Hsü and A. Hsü showed in [2] that the music pieces they studied satisfy the relation

$$F = \frac{c}{I^D} \tag{1}$$

or equivalently, taking the natural log for (1) gives the linear log-log model

$$\ln(F) = \ln(c) - D\ln(I). \tag{2}$$

Using least squares for data points $[\ln(F), \ln(I)]$, one can find the values of $\ln(c)$ and D, and hence a model of the form of equation (1).

In our study, the interval I of length 0 occurred quite often. Since the natural log is not defined at 0, we instead consider the model

$$F = \frac{c}{(I+1)^D}. \tag{3}$$

In a similar way, we examine the log-log model

$$\ln(F) = \ln(c) - D\ln(I+1). \tag{4}$$

Notice that $\ln(F)$ is defined as we ignored intervals with zero frequency. Using the LeastSquare command in Maple, we find the equation and graph of the best fit line $Y = mX + b$. Here $Y = \ln F$ and $X = \ln(I+1)$. Hence we find optimal values for c and D. We also find the correlation coefficient ρ for our data to make sure that the least squares line is a reasonable representation of our data. The graphs, linear equations and correlations for each music piece studied can be found in Table 1 in Appendix .

For the eight music pieces we find the following results:

- From Tone 1, we studied "Lord I Have Cried" [7]. One can listen to this piece at [8]. The data satisfies the model

  $$F = \frac{e^{3.851021646}}{(I+1)^{1.449698513}}.$$

  The data correlation coefficient is $\rho = -0.798727790492442$.

- From Tone 2, we studied "Athimi Ya Nafsi" [10]. The data satisfies the model

  $$F = \frac{e^{3.741703038}}{(I+1)^{1.123857777}}.$$

  The data correlation coefficient is $\rho = -0.609670479009270$.

- From Tone 3, we studied "Ayuha Al Mukhalles" [6]. The data satisfies the model

  $$F = \frac{e^{3.913596659}}{(I+1)^{1.378555537}}.$$

  The data correlation coefficient is $\rho = -0.779620236668013$.

- From Tone 4, we studied "Dogmatic Theotokion" [13]. The data satisfies the model

  $$F = \frac{e^{4.349722737}}{(I+1)^{2.437787120}}.$$

  The data correlation coefficient is $\rho = -0.877945272395581$.

- From Tone 5, we studied "Awed by thy Beauty" [15]. One can listen to this piece at [14]. The data satisfies the model

  $$F = \frac{e^{4.206943955}}{(I+1)^{2.373352327}}.$$

  The data correlation coefficient is $\rho = -0.843075302030584$.

- From Tone 6, we studied another "Dogmatic Theotokion" [16]. One can listen to this piece at [17]. The data satisfies the model

$$F = \frac{e^{4.022033080}}{(I+1)^{2.139028887}}.$$

The data correlation coefficient is $\rho = -0.842805097034332$.

- From Tone 7, we studied "Inna Hirodos" [9]. One can listen to this piece at [12]. The data satisfies the model

$$F = \frac{e^{4.112762146}}{(I+1)^{2.282566212}}.$$

The data correlation coefficient is $\rho = -0.785737876049626$.

- From Tone 8, we studied "It is Truly Right" [18]. One can listen to this piece at [19]. The data satisfies the model

$$F = \frac{e^{3.853714954}}{(I+1)^{1.713851239}}.$$

The data correlation coefficient is $\rho = -0.770719426555143$.

Conclusion

By studying 8 representative songs, one from each of the 8 Tones in Byzantine music and converting each note to its corresponding semitone frequency, we have demonstrated that the 8 songs we studied exhibit a relation in the form of equation (3), and are strongly correlated. We have only explored a very small sample of the songs found in Byzantine music, but we conjecture that all the songs in Byzantine music will exhibit a similar model in equation (3), because the music is composed monophonically and each one of the eight Tones has its own pattern and rules that it must follow.

Acknowledgements

We would like to thank the following people and organizations:

- David Koelle for developing the JFugue software used in our project and answering our Java questions.

- Father Ephraim and the individuals at St. Anthony's Monastery for answering questions and their work on the Divine Music Project which made the Byzantine Music sheets available to us. There are few sources available in English for Byzantine music. Many are written in Greek and Arabic. The individuals at St. Anthony's Monastery are experts in Byzantine music. They have worked hard on translating many of the original Greek Byzantine music pieces into English while preserving the Byzantine style.

- Nichoals Ayoub, the Protopsaltis at St. Nicholas Antiochian Orthodox Church, for providing Byzantine music support and proofreading our collected data.

Appendix

Tone 7, “Inna Hirodos” [9]

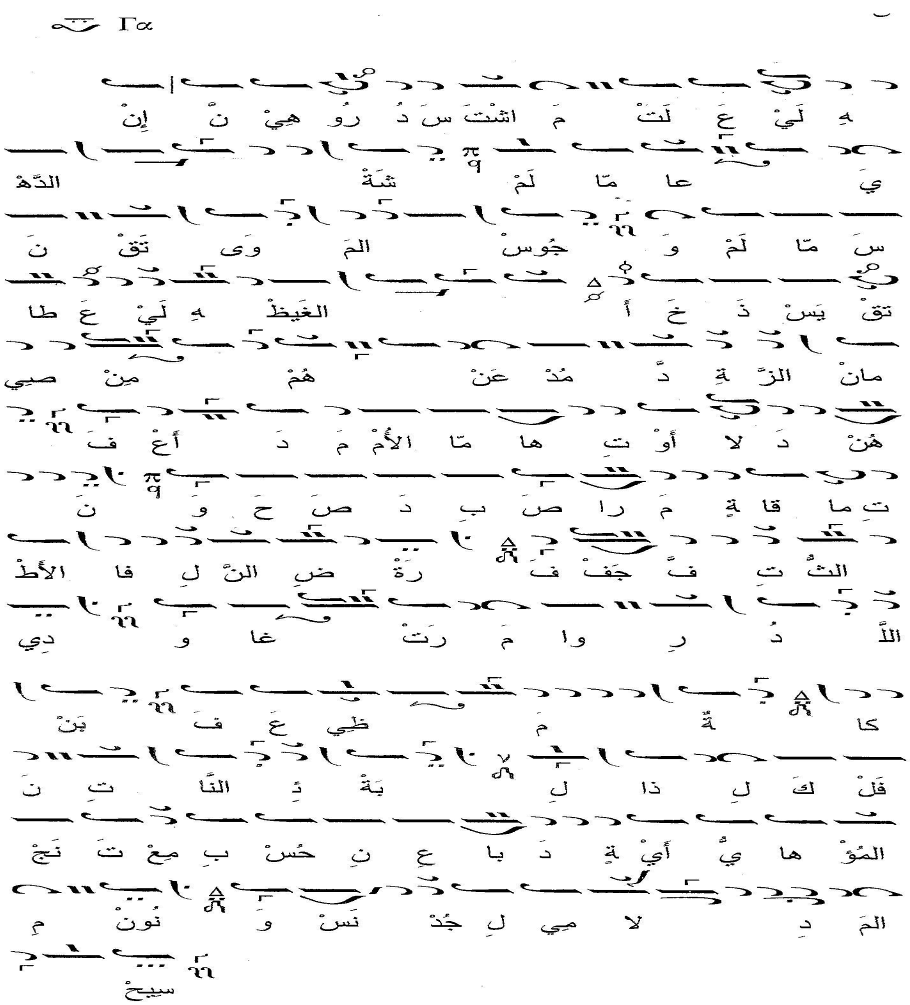
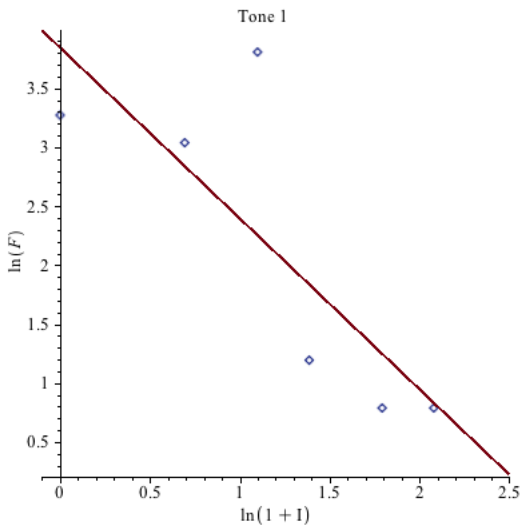
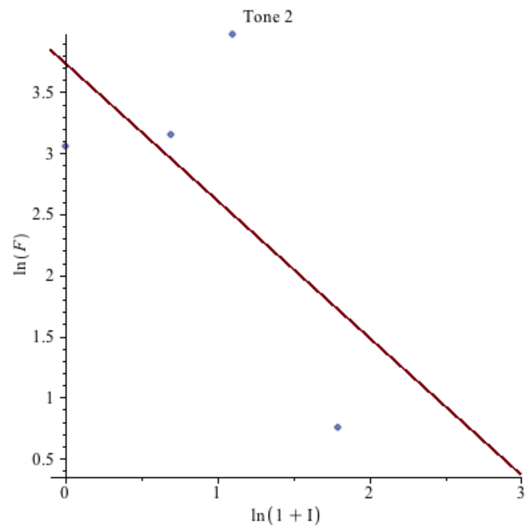


Figure 3: A representative of Tone 7, “Inna Hirodos” from [9]

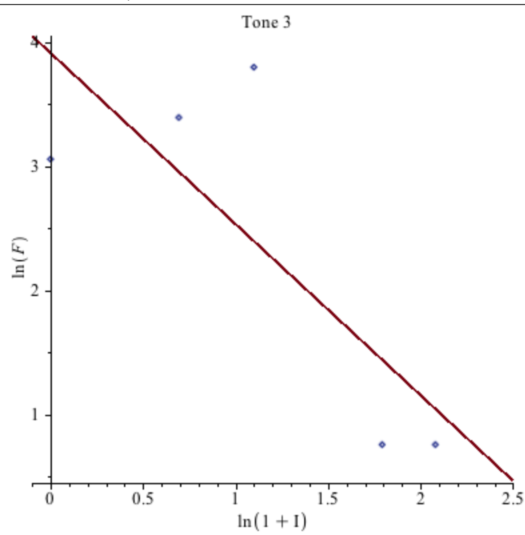
The results for the 8 Tones



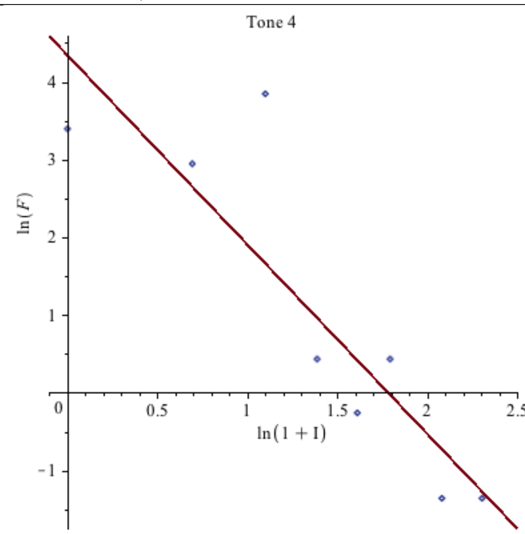
$Y = 3.851021646 - 1.449698513\,X$
$\rho = -0.798727790492442$



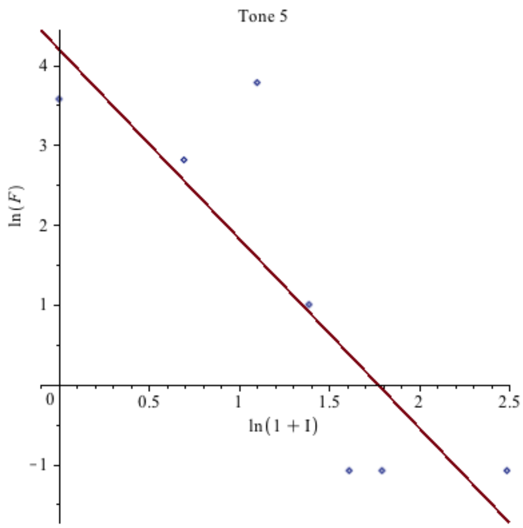
$Y = 3.741703038 - 1.123857777\,X$
$\rho = 0.609670479009270$



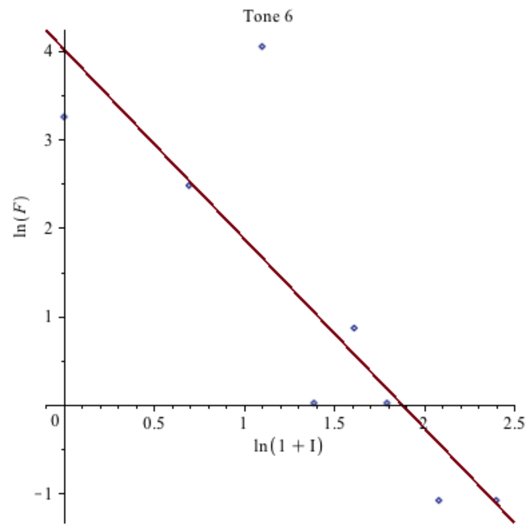
$Y = 3.913596659 - 1.378555537\,X$
$\rho = -0.779620236668013$



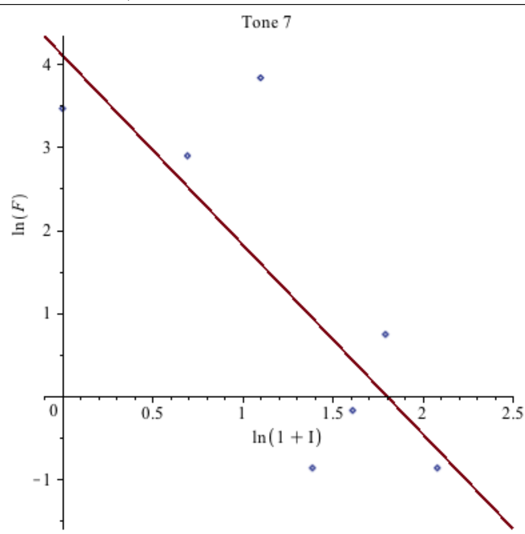
$Y = 4.349722737 - 2.437787120\,X$
$\rho = 0.877945272395581$



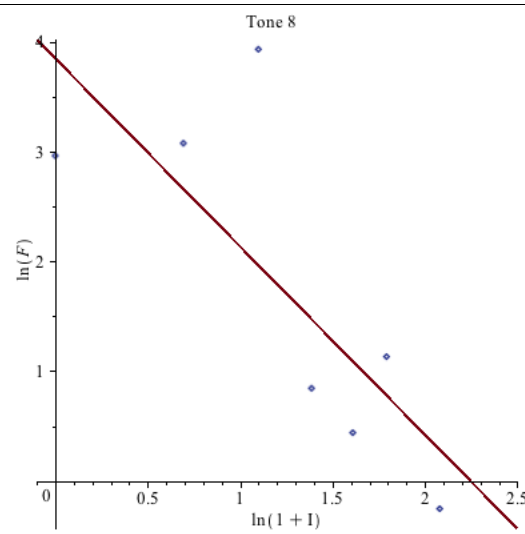
$Y = 4.206943955 - 2.373352327\,X$
$\rho = -0.843075302030584$



$Y = 4.022033080 - 2.139028887\,X$
$\rho = -0.842805097034332$



$Y = 4.112762146 - 2.282566212\,X$
$\rho = -0.785737876049626$



$Y = 3.853714954 - 1.713851239\,X$
$\rho = -0.770719426555143$

Table 1: Graphs for the best fit log-log line for Tones 1-8

Semitone Frequencies

Note	Frequency (Hz)	Semitone Frequency
$N\eta$	261.6	60
$P\alpha$	293.7	62
Bov	329.6	64
$\Gamma\alpha$	349.2	65
$\Delta\iota$	392	67
$K\varepsilon$	440	69
$Z\omega$	493.9	71
$N\eta$	523.3	72

Table 2: Frequencies in Hertz and semitone frequencies for notes in Byzantine music [3].

Example from Tone 7

(I)	f_I	F_I	$\ln(I+1)$	$\ln(F_I)$
0	76	31.93277311	0	35.39640596
1	43	18.06722689	0.693147181	20.96132652
2	110	46.21848739	1.098612289	50.05186727
3	1	0.420168067	1.386294361	-0.44693242
4	2	0.840336134	1.609437912	0.666382827
5	5	2.100840336	1.791759469	2.843177761
7	1	0.420168067	2.079441542	-0.44693242
Total	238	100		

Table 3: Results for the representative song from Tone 7

References

[1] D. Conomos, A Brief Survey of the History of Byzantine and Post-Byzantine Chant. Retrieved June 30, 2009, from St. Anthony's Orthodox Monastery website: `http://bit.ly/TXXu88`.

[2] K. Hsü, A. Hsü,*Fractal Geometry of Music*. Proceedings of the National Academy of Sciences of the United States of America. **87** (1990) 938–941.

[3] D. Koelle, The Complete Guide to JFugue: Programming Music in Java™. Retrieved from `http://www.jfugue.org/`.

[4] F. Hindeleh, J. Sears, L. Copenhaver, Table of data for Tone. Retrieved from `http://bit.ly/Y3Xs39`

[5] Indiana University School of Music Center for Electronic and Computer Music. (August 31, 2003). *MIDI Note Number to Equal Temperament Semitone to Hertz Conversion Table*. Retrieved from: `http://bit.ly/UzY1Od`.

[6] M. El Murr, Holy Week, Al Ossbou Al Muqaddas (Arabic). Beirut, Lebanon. pp.116–117.

[7] St. Anthony's Orthodox Monastery, The Divine Music Project. Retrieved July 24, 2009, from St. Anthony's Orthodox Monastery' website: `http://bit.ly/WmDpdX`.

[8] St. Anthony's Orthodox Monastery, The Divine Music Project. Retrieved July 24, 2009, from St. Anthony's Orthodox Monastery' website: `http://bit.ly/Y7ts8K`.

[9] M. El Murr, Spiritual Pipe: Part 1, Al Mizmar Al Rohi: Al Juzo' Al Awal (Arabic). Beirut, Lebanon. pp 125, 1972.

[10] M. El Murr, Spiritual Pipe: Part 1, Al Mizmar Al Rohi: Al Juzo' Al Awal (Arabic). Beirut, Lebanon. pp. 225, 1972

[11] Byzantine Music History. P. Papadakis; [cited September 8, 2009]. Available from: `http://bit.ly/VO1bgk`.

[12] Serafemsarof; Inna Hirodos. Uploaded December 11, 2010 on Youtube `http://bit.ly/X06GPC`.

[13] St. Anthony's Orthodox Monastery, The Divine Music Project. Retrieved July 24, 2009, from St. Anthony's Orthodox Monastery' website: http://bit.ly/Yhnn9h.

[14] St. Anthony's Orthodox Monastery, The Divine Music Project. Retrieved July 24, 2009, from St. Anthony's Orthodox Monastery' website: http://bit.ly/10cJQVW.

[15] St. Anthony's Orthodox Monastery, The Divine Music Project. Retrieved July 24, 2009, from St. Anthony's Orthodox Monastery' website: http://bit.ly/WnYBUG.

[16] St. Anthony's Orthodox Monastery, The Divine Music Project. Retrieved July 24, 2009, from St. Anthony's Orthodox Monastery' website: http://bit.ly/T69GGL.

[17] St. Anthony's Orthodox Monastery, The Divine Music Project. Retrieved July 24, 2009, from St. Anthony's Orthodox Monastery' website: http://bit.ly/13EoBvc.

[18] St. Anthony's Orthodox Monastery, The Divine Music Project. Retrieved July 24, 2009, from St. Anthony's Orthodox Monastery' website: http://bit.ly/V3iOYT.

[19] St. Anthony's Orthodox Monastery, The Divine Music Project. Retrieved July 24, 2009, from St. Anthony's Orthodox Monastery' website: http://bit.ly/VKJ2ox.

[20] St. Anthony's Orthodox Monastery, The Divine Music Project. Retrieved September 17, 2009, from St. Anthony's Orthodox Monastery' website: http://bit.ly/YhoHZK.

[21] Y. Yazigi, Foundations of Byzantine Music, St. John of Damascus Institute of Theology, El Koura, Lebanon, 2001.