## Extended Thesis Abstracts

# The Finite Radon Transform

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## Introduction

This thesis explores a finite version of the *Radon transform*, an integral transform achieved by integrating a function over a set of lines in its domain (see [6]). We define the *finite Radon transform* $R : C(X) \to C(Y)$ as:

$$Rf(y) = \sum_{x \in y} f(x),$$

where $X$ is a finite set, $Y$ is a collection of subsets of $X$, and $C(X)$ and $C(Y)$ denote complex-valued functions on $X$ and $Y$, respectively. The main purpose of this thesis is to investigate the injectivity and bijectivity of this transform.

The (continuous) Radon transform has found application in *tomography*, whose objective is to see what is inside an object without opening it up, e.g., the interior structures of the human body, rocket motors, rocks, snow packs on the Alps, and violins (see [5]). The tomography objective is similar to a classic problem in mathematics: determine an unknown using given information. In tomography, the given information is a set of $x$-ray projections of some unknown object, and the solution is a representation of the unknown object. Mathematically, the collection of $x$-ray projections of an object through all possible lines is analogous to an image $Rf$ of the Radon transform, where $f$ is an unknown density function defined on the interior of the object. The inverse problem of determining an object via its $x$-ray projections can be described as: Given a Radon image $Rf$, when can we invert $R$ to recover $f$?

In order to answer this question we would like to determine when $R$ is injective and when $R$ is bijective. This thesis investigates conditions under which the finite Radon transform is injective, and then, for certain instances of the Radon transform (the $k$-set transform and the affine $k$-plane transform), presents conditions for bijectivity. In other words, we want to describe situations where every element in the co-domain $C(Y)$ of the finite Radon transform can be used to recover an element of the domain. Since the Radon transform is linear, one can use basic linear algebra to recover $f$ from $Rf$ in the case that R is injective.

## Examples and Point-Line Incidence Matrices

In this section we present some specific examples of the finite Radon transform and we introduce their matrix representations.

### The $k$-set transform

Let $X = \{x_1, \ldots, x_n\}$ be a finite set and $Y$ be the set of all subsets of $X$ that contain exactly $k$ elements. The corresponding Radon transform $R : C(X) \to C(Y)$ is called the $k$-set transform. When $k = 2$ we can identify the pair $(X, Y)$ with the complete graph $K_n$ on $n$ vertices.

**Example 1.** *For the set $X = \{x_1, x_2, x_3, x_4\}$ we have $Y = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}\}$. Note that $X$ is the set of vertices and $Y$ is the set of edges of the graph $K_4$ shown in Figure 1.*

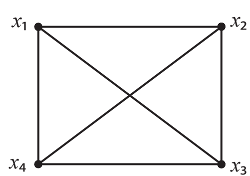


Figure 1: $K_4$

The connection with graphs allows us to express results about the 2-set transform in graph theoretic terms and to use knowledge about graph theory (e.g., counting $p$-trees and cycles) to obtain the results that will be stated later.

### The affine $k$-plane transform

Let $F$ denote a finite field, $X = F^d = \{(a_1, a_2, \ldots, a_d) |\, a_j \in F\}$ a $d$-dimensional vector space over $F$, and $Y$ denote the set of all affine $k$-dimensional spaces in $X$, i.e., $Y = \{\overrightarrow{a} + V : \overrightarrow{a} \in X \text{ and } V \text{ is a } k\text{-dimensional subspace of } X\}$. The corresponding finite Radon transform $R : C(X) \to C(Y)$ is called the affine $k$-plane transform.

**Example 2.** *If $F = \mathbb{Z}_3$, $d = 2$, and $k = 1$, we get $X = \mathbb{Z}_3^2 = \{(0,0),(1,0),(2,0),(0,1),(1,1),(2,1),(0,2),(1,2),(2,2)\}$ and $Y = \{y_1, y_2, \ldots, y_{12}\}$, where*

$$\begin{array}{ll} y_1 = \{(0,0),(1,0),(2,0)\}, & y_7 = (1,0) + y_2 \\ y_2 = \{(0,0),(0,1),(0,2)\}, & y_8 = (2,0) + y_2 \\ y_3 = \{(0,0),(1,1),(2,2)\}, & y_9 = (0,1) + y_3 \\ y_4 = \{(0,0),(2,1),(1,2)\}, & y_{10} = (0,2) + y_3 \\ y_5 = (0,1) + y_1, & y_{11} = (0,1) + y_4 \\ y_6 = (0,2) + y_1, & y_{12} = (0,2) + y_4. \end{array}$$

*To illustrate the transform, we consider $f$ defined by $f((i,j)) = i + j$ and see $Rf(y_3) = 0 + 2 + 4 = 6$.*

We are particularly interested in the case where $Y$ consists of hyperplanes in $X$ because we can present conditions guaranteeing bijectivity of the finite Radon transform in this special case (see Theorem 6).

**Point line incidence matrices**

The definition of the finite Radon transform implies that the transform is linear. We now define a matrix that can be used to represent the transform.

**Definition 1.** *When $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ is a collection of subsets of $X$, for $1 \le i \le m$ and $1 \le j \le n$, put*

$$a_{ij} = \begin{cases} 0 & x_j \notin y_i \\ 1 & x_j \in y_i \end{cases}.$$

*The $m \times n$ matrix $A = (a_{ij})$ is called the point-line incidence matrix.*

Point-line incidence matrices corresponding to the specific sets $X$ and $Y$ given in Example 1 and Example 2 above are

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

respectively.

## Injectivity and Block Conditions

We now introduce conditions under which the finite Radon transform is injective. Recall that injectivity of the Radon transform is important because it allows us to reconstruct a function $f$ from an image element $Rf$.

First set $G_x = \{y \in Y | x \in y\}$. In other words, $G_x$ is the set of all elements of $Y$ that contain a fixed $x$. Given $X$ and $Y$, Bolker [1] declares the 'block' conditions to be satisfied if there exist positive numbers $\alpha$ and $\beta$ such that $\alpha \neq \beta$ and

$$|G_x| = \alpha \quad \forall x \in X$$
$$|G_x \cap G_{x'}| = \beta \quad \forall x, x' \in X \text{ with } x \neq x'.$$

Observe that if the block conditions are in place then $\alpha$ is the sum of the entries in any given column of the corresponding point-line incidence matrix while $\beta$ is the dot product of any two distinct columns.

**Theorem 2** ([1]). *When the block conditions are satisfied, the corresponding Radon transform $R$ is injective.*

**Theorem 3** ([1]). *If $X$ admits a doubly transitive group action which induces a group action on $Y$, then the block conditions are satisfied.*

In this thesis, Theorem 2 is proved by showing that $A^T A$ is invertible, where $A$ is the corresponding point-line incidence matrix. This is a different method than Bolker uses in [1]. Further, one can show that both the $k$-set and affine $k$-plane transforms admit doubly transitive group actions, yielding the following corollary:

**Corollary 4** ([1]). *The $k$-set transform, as well as the affine and projective $k$-plane transforms, are injective.*

## Admissibility

Now we go a step further and ask: When is $R$ both one-to-one and onto? In order for an injective $R : C(X) \to C(Y)$ to be bijective, we need the vector spaces $C(X)$ and $C(Y)$ to have the same dimension. Therefore, if $|X| = n$, then $|Y|$ must be $n$ as well. To meet this condition we throw out some elements of $Y$ (which is often larger than $X$) to obtain $Y' \subseteq Y$ with $|Y'| = |X|$. However, one cannot simply throw out arbitrary elements because it is possible that the resulting transform will no longer be injective. In this section we show, in some special cases, how to choose subsets of $Y$ so that the corresponding finite Radon transform is bijective. Such $Y'$ are called *admissible* subsets of $Y$. Note that when $Y'$ is admissible, every element of $C(Y')$ is $Rf$ for some $f \in C(X)$, so $f$ can be reconstructed from $Rf$.

### The affine k-plane transform and admissibility

In order to determine what subsets of $Y$ are admissible for the affine $k$-plane transform, we first need to define a spread.

**Definition 5.** *A spread is a subset of $Y$ that determines a partition of $X$.*

**Example 3.** *If $X = \mathbb{Z}_3^2$ and $Y = \{y_1, y_2, \ldots, y_12\}$ are as in the Example 2, then each of $\{y_1, y_5, y_6\}, \{y_2, y_7, y_8\}, \{y_3, y_9, y_{10}\}$, and $\{y_4, y_{11}, y_{12}\}$ are spreads. All of these spreads are formed by translates of a 1-dimensional subspace.*

**Theorem 6** ([3]). *Let $R : C(X) \to C(Y)$ be an affine $k$-plane transform with $k = d - 1$, and let $\{w_1, w_2, \ldots, w_k\}$ be the set of spreads in $Y$. The set $\{w_1, \widehat{w}_2, \widehat{w}_3, \ldots, \widehat{w}_k\}$ is an admissible subset of $Y$, where $\widehat{w}_j$ is formed by casting out one element from $w_j$ for each $j$ with $2 \le j \le k$. All admissible subsets of $Y$ are formed in this way.*

An application of Theorem 6 shows that $Y' = \{y_1, y_5, y_6, y_2, y_7, y_3, y_9, y_4, y_{11}\}$ is an admissible set for the affine $k$-plane transform determined by $X = \mathbb{Z}_3^2$ and $Y = \{y_1, y_2, \ldots, y_{12}\}$ as in Example 2.

A key step in the proof of Theorem 6 is to show that if $Y'$ is formed by casting out hyperplanes as described in the theorem, then the corresponding point-line matrix $A'$ is a square matrix and the rank of $A'$ equals the rank of $A$. But by Corollary 4 we know that $A$ has full rank, so we are able to conclude that $A'$ has full rank and hence $Y'$ is admissible.

When $k \neq d-1$, characterizing admissible subsets for the $k$-plane transform is an open problem.

### The 2-set transform and admissibility

In the $k$-set transform $R : C(X) \to C(Y)$ when $k = 2$, we recall that $(X, Y)$ can be identified with the complete graph $K_n$ on $n$ vertices, where $|X| = n$. In order to characterize admissible subsets of $Y$, we first introduce terminology that will be used when describing complete graphs.

Let $Y'$ be a subset of $Y$. A *$Y'$-path* is a sequence of points in $X$ such that consecutive points are joined by an edge in $Y'$. A $Y'$ path is a *$Y'$-cycle* if it begins and ends at the same point. A point $x \in X$ lies on an *odd $Y'$-cycle* if there is a $Y'$-cycle $\alpha$ such that

(i) $\alpha$ begins and ends with $x$, and

(ii) $\alpha$ has an even number of points (connected by an odd number of edges).

Two $Y'$-cycles are equivalent if their edges form the same set. For example, let $X = \{x_1, \ldots, x_4\}$ and $Y = \{y_1, \ldots, y_6\}$ as shown in Figure 2, with $Y' = \{y_1, y_2, y_4, y_5\}$. An example of a $Y'$-path is $(x_1, x_2, x_3, x_2)$ and an odd $Y'$-cycle is $(x_3, x_2, x_4, x_1, x_2, x_3)$.

**Theorem 7** ([4]). *Let $(X, Y) = K_n$ and with $|Y| = |X|$. The set $Y'$ is admissible for the corresponding 2-set transform if and only if each $x \in X$ lies on an odd cycle.*

Knowing how to form admissible subsets in the 2-set transform, one may ask how many such admissible subsets exist. Results from graph theory about

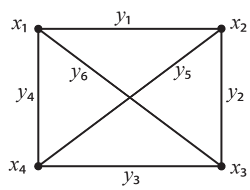


Figure 2: A $Y'$-path and an odd $Y'$-cycle

$p$-trees found in [2] are used to develop a counting method. This counting method is explored further in this thesis. Below is a table of values of numbers of admissible subsets given $|X| = n$:

| $n$ | Number of admissible subsets |
|---|---|
| 3 | 1 |
| 4 | 12 |
| 5 | 162 |
| 6 | 2530 |
| 7 | 45615 |
| 8 | 937440 |
| 9 | $2.1685135 \times 10^7$ |
| 10 | $5.58360144 \times 10^8$ |
| 11 | $1.5850436805 \times 10^{10}$ |
| 12 | $4.9202966692 \times 10^{11}$ |
| 13 | $1.658522958675 \times 10^{13}$ |
| 14 | $6.03402126941952 \times 10^{14}$ |

# References

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