The Oberwolfach Problem in Graph Theory

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During the summer of 2007, I participated in a Research Experience for Undergraduates (REU) at Illinois State University. I was one of eight undergraduates, all of whom were mathematics education majors. During my eight week experience, we worked with four in-service teachers from the Normal-Bloomington area, as well as math and math education professors from Illinois State and surrounding universities. We discussed several applications of discrete mathematics and did original research, particularly in graph theory. In this article, I will be discussing the *Oberwolfach problem*, an open problem in graph theory that I spent most of my time working on throughout the REU.

Graph Theory Concepts

In this section, we will review concepts that will be needed later to understand the Oberwolfach problem. A *simple graph* G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a nonempty finite set and $E(G)$ is a set of 2-element subsets of G. The elements of $V(G)$ are called *vertices*, and the elements of $E(G)$ are called *edges.* The number of vertices in a graph is called the *order* of the graph and the number of edges in a graph is called the *size* of the graph. Figure 1 is an example of a graph of order 4 and size 5, where the vertices are represented by points and the edges are lines connecting those points.

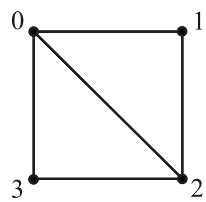

Figure 1: A graph of order 4 and size 5.

Two vertices are said to be *adjacent* if there is an edge connecting those two vertices. In Figure 1, vertex 1 is adjacent to vertices 0 and 2. Two graphs G and H are said to be *isomorphic* if there exists a one-to-one and onto map from $V(G)$ to $V(H)$ that preserves adjacency. Two isomorphic graphs may appear dissimilar, but are intrinsically equivalent.

The *degree* of a vertex is the number of edges incident with that vertex. For example, in Figure 1, vertex 0 has degree 3 and vertex 1 has degree 2.

The graph of order n, where every vertex is adjacent to every other vertex, is called the *complete graph* on n vertices and is denoted by K_n. In K_n, there are $n(n-1)/2$ edges. Figure 2 shows an example of a complete graph on 7 vertices. In a complete graph, every vertex has degree $(n-1)$.

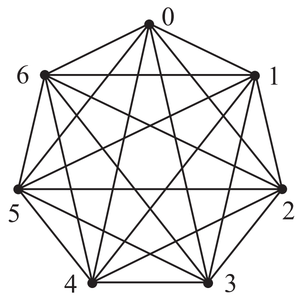

Figure 2: K_7

A *cycle* is a closed path with no repetitions. A cycle with n vertices has n edges and is denoted by C_n. In a cycle, every vertex has degree 2. Figure 3 shows an example of a C_5.

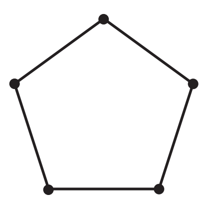

Figure 3: C_5

Label the vertices in K_n so that $V(K_n) = \{0, 1, \ldots, n-1\}$. The *length* of an edge $\{i, j\}$ is defined to be $\min\{|i-j|, n-|i-j|\}$. Intuitively, the length of $\{i, j\}$ is the number of edges needed to join i to j in C_n, where vertex m is always adjacent to vertex $m+1$ modulo n. In Figure 4, the edge connecting vertices 0 and 5 has length 1 and the edge connecting vertices 0 and 3 has length 3.

The number of edges of a given length in K_n varies depending on the parity of n. If $n = 2t+1$, the longest edge length will be t, and K_n will have n edges of length k for $k = 1, 2, \ldots, t$. On the other hand, if $n = 2t$, then the longest edge length is t and K_n will have n edges of length k for $k = 1, 2, \ldots, t-1$ together with t edges of length t.

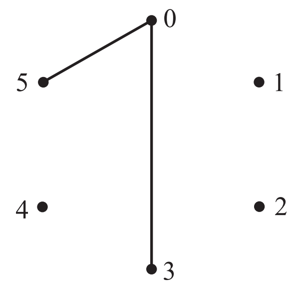

Figure 4: Edge $\{0,5\}$ has length 1, whereas edge $\{0,3\}$ has length 3.

Clicking of an edge $\{i,j\}$ modulo m is understood to be the increase of both i and j by 1, where addition is taken modulo m. Clicking of a graph G modulo m is the simultaneous clicking of all edges of G modulo m. Observe that clicking preserves edge length.

Through repeated clicking, larger complete graphs can be obtained from smaller graphs. In Figure 5, it is possible to reproduce a K_7 be clicking the dotted C_3 six times modulo 7.

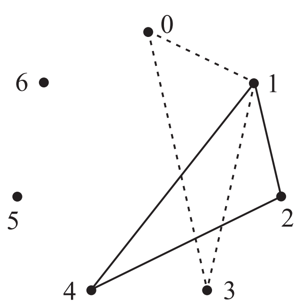

Figure 5: Repeated clicking of the dotted C_3, modulo 7, results in a K_7.

The Oberwolfach Problem

The Oberwolfach Problem was first posed in 1967, when several mathematicians interested in graph theory gathered in Oberwolfach, Germany for a conference. The question is whether it is possible to seat an odd number m of mathematicians at n round tables in $(m-1)/2$ meals so that each mathematician sits next to everyone else exactly once. If the n round tables are of sizes $k_1, k_2, \ldots, k_n$ (with $k_1 + k_2 + \cdots + k_n = m$) then we denote the corresponding Oberwolfach problem as $OP(k_1, k_2, \ldots, k_n)$.

For example, when scheduling a conference with 11 mathematicians seated at one table of three and two tables of four, a solution to $OP(3,4,4)$ looks as follows, where the mathematicians are labeled $0, 1, \ldots, 10$:

Meal 1	Meal 2	Meal 3	Meal 4	Meal 5
0, 10, 5	1, 10, 6	2, 10, 7	3, 10, 8	4, 10, 9
1, 4, 6, 9	2, 5, 7, 0	3, 6, 8, 1	3, 7, 9, 2	5, 8, 0, 3
8, 7, 3, 2	9, 8, 4, 3	0, 9, 5, 4	1, 0, 6, 5	2, 1, 7, 6

Graph theory can be applied to the Oberwolfach problem. One can think of each "table" as a cycle, and "people" are thought of as vertices. Two people sitting next to each other is described as an edge existing between two vertices. The Oberwolfach problem then corresponds to decompositions of complete graphs using cycles: every person sitting next to every other person corresponds to there being an edge between every pair of two vertices. The decompositions must be vertex disjoint, meaning that a person can only sit at one table per meal.

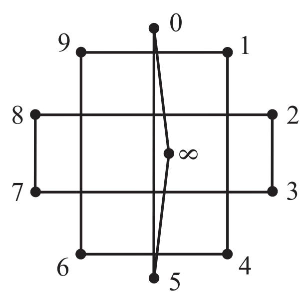

Figure 6: Seat arrangement for Meal 1 in our solution to $OP(3, 4, 4)$.

For example, let's consider $OP(3, 4, 4)$. In Figure 6, one "table" in the seating arrangement would be the C_4 containing the vertices 9, 1, 4, and 6. During this "meal," person 9 would sit next to persons 1 and 6. Note that we have moved one vertex into in the middle of the figure and relabeled it ∞. This creates a fixed point for the clicking operation modulo 10, which now may be thought of as a rotation centered at ∞. New seating arrangements are obtained by clicking (rotating) the arrangement in Figure 6 modulo 10, with repetition of seating arrangements beginning with the fifth click. Observe that these seating arrangements form the above solution to $OP(3, 4, 4)$:

Meal 1	Meal 2	Meal 3	Meal 4	Meal 5
0, ∞, 5	1, ∞, 6	2, ∞, 7	3, ∞, 8	4, ∞, 9
1, 4, 6, 9	2, 5, 7, 0	3, 6, 8, 1	3, 7, 9, 2	5, 8, 0, 3
8, 7, 3, 2	9, 8, 4, 3	0, 9, 5, 4	1, 0, 6, 5	2, 1, 7, 6

For larger cases, pictures like Figure 6 become complicated and cumbersome, so we label the cycles separately, as shown in Figure 7. The edge lengths are also recorded in these drawings, and makes possible solutions to the Oberwolfach problem more clear: Notice that each edge length $1, 2, 3, 4$ and ∞ must occur twice in the initial seating arrangement and 5 must occur once, because K_{11} consists of edges of precisely these (modified) lengths and our clicking operations preserves edge lengths.

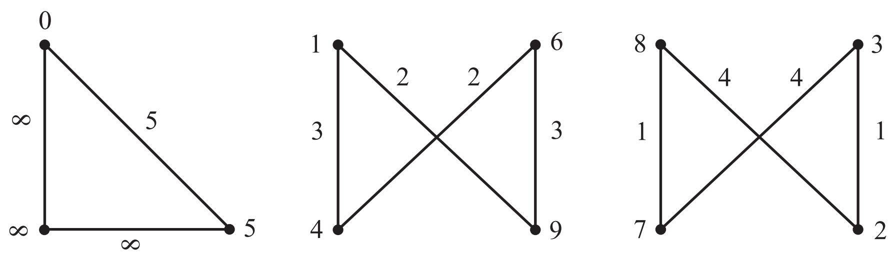

Figure 7: The cycles of Figure 6.

In our research this summer, we studied several known patterns that form solutions to the Oberwolfach problem. For example, Figures 8 and 9 show solutions to $OP(3, 3, 4t+3)$ in case $t = 1$ and $t = 2$. Complete seating arrangements can be found by clicking these initial seating arrangements modulo 12 and modulo 16, respectively.

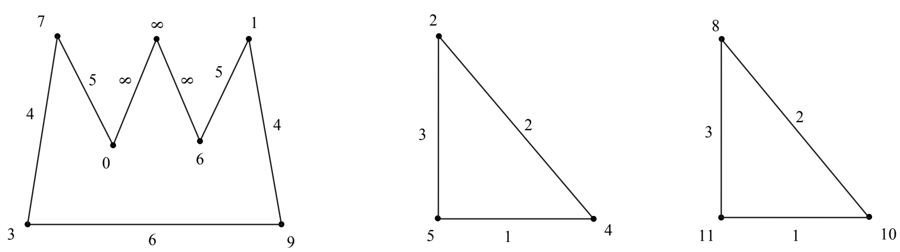

Figure 8: Decomposition of a K_{13} using C_7, C_3 and C_3, respectively.

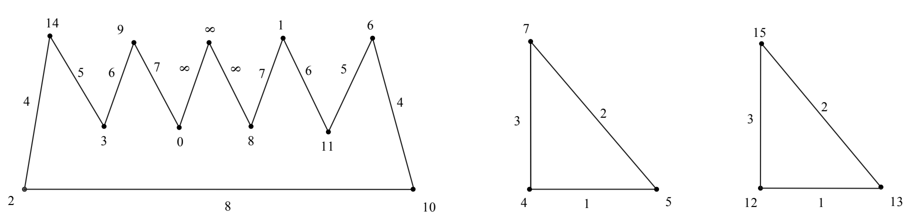

Figure 9: Decomposition of a K_{17} using C_{11}, C_3 and C_3, respectively

This pattern continues, where edge lengths 1, 2, and 3 are in the C_3's and the rest of the lengths are in the C_{4t+3}, where the longest finite length is placed on the bottom edge and the remaining lengths begin at 4 and increase as one moves toward the vertex ∞. This provides a solution to $OP(3,3,4t+3)$ for $t \geqslant 1$. Observe that the C_{4t+3} is self-symmetric when looking at the edge lengths.

Some of the original research that we have done involves trying to find a similar solution for $OP(5,5,4t+1)$ for $t \geqslant 2$. There is a pattern that seems to be developing; however, it will not work for $t > 4$. In particular, the edge length immediately after the edge of length 1 seems to increase by one for each unit increase of t, but since the C_5's only contain 5 different lengths, there will be no way to include all of the remaining smaller lengths in the C_5's. The first two cases of this situation are shown in Figures 10 and 11:

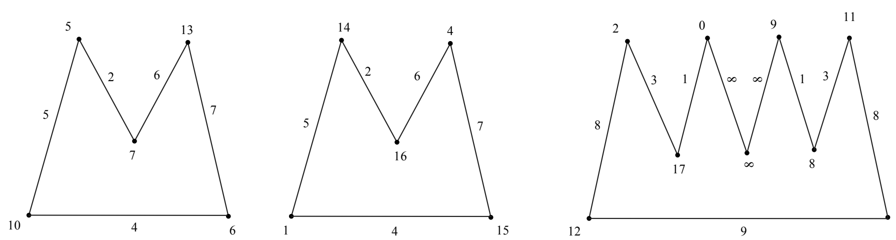

Figure 10: Decomposition of a K_{19} using C_5, C_5 and C_9, respectively

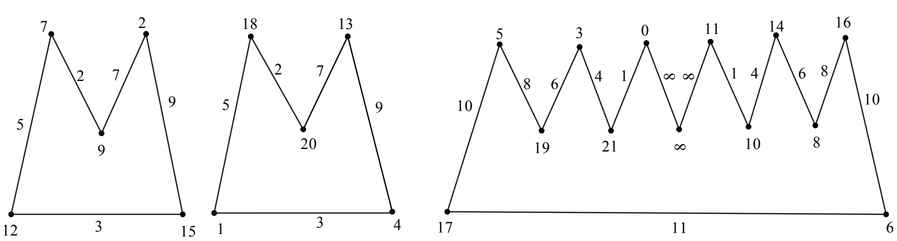

Figure 11: Decomposition of K_{23} using C_5, C_5 and C_{13}, respectively

My current work includes trying to solve $OP(5,5,4t+1)$ for $t \geqslant 2$ by extending the arrangements I have found for $t = 2, 3, 4$. If no pattern seems to exist, I might try C_{4t+1}, C_{4t+1}, and C_{8t+1}. From these introductory examples, there are many directions one could take, and my continued research could find success in one of those directions.

Many cases of the Oberwolfach problem have already been solved. In order to list them efficiently, it is customary to abbreviate repeated cycles as powers.

For example, $OP(5,5,9)$ would simply be recorded as $OP(5^2,9)$. According to [1], other than $OP(3,3)$, $OP(3,3,3,3)$, $OP(4,5)$, and $OP(3,3,5)$, none of which has a solution, the following Oberwolfach problems all have solutions that have been found:

1. $OP(m^t)$ for all $t \geqslant 1$ and $m \geqslant 3$;
2. All cases where the total cycle length sum is less than or equal to 17 (for example, C_4, C_4, and C_7 in a decomposition of K_{15});
3. $OP(3^k, 4)$ for all odd $k \geqslant 1$;
4. $OP(3^k, 5)$ for all even $k \geqslant 4$;
5. $OP(r^k, n-kr)$ for $n \geqslant 6kr-1$, $k \geqslant 1$, $r \geqslant 3$;
6. $OP(r, n-r)$ for $r = 3,4,5,6,7,8,9$ and $n \geqslant r+3$;
7. $OP(r, r, n-2r)$ for $r = 3,4$ and $n \geqslant 2r+3$;
8. $OP(2r_1, 2r_2, \ldots, 2r_k)$ for all $r_i \geqslant 2$ and $r_1 + r_2 + \cdots + r_k$ odd;
9. $OP(r, r+1)$ and $OP(r, r+2)$ for $r \geqslant 3$;
10. $OP(2s+1, 2s+1, 2s+2)$ for $s \geqslant 1$;
11. $OP(3, 4s, 4s)$ for $s \geqslant 1$;
12. $OP(4^\alpha, 2s+1)$ for $s \geqslant 1$, $\alpha \geqslant 0$;
13. $OP((4s)^\alpha, 2s+1)$ for $s \geqslant 1$, $\alpha \geqslant 0$.

References

[1] C. Colbourn and J. Dinitz (Editors), Handbook of Combinatorial Designs, Discrete Mathematics and its Applications (Boca Raton), CRC Press, 2006.