Articles

Turning Tables, Slicing Pizza, and the Brouwer Fixed-Point Theorem

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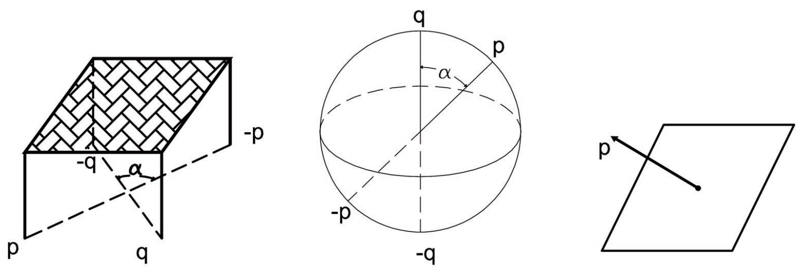
Mathematics is everywhere in life. Even within the short dinner time, it helps me solve two big problems.

Scene 1: I have confidence in saying that the four legs of my kitchen table have the same length, since it cost me a lot of money. Unfortunately, it wobbles because of my old floor, which I cannot afford to fix right now. Fortunately, the Dyson-Livesay Theorem gives me a cheaper solution. It tells me that I can fix this by just rotating the table by some angle.

Connect the four feet of our rectangular table diagonally with two line segments. Then these two segments intersect at some angle α and form two diameters of some sphere S^2. (See Figure 1(a), 1(b).) Imagine lifting the table, along with the sphere, high above the ground and let $f(x)$ denote the vertical distance from a point x on that sphere to the floor. This function is clearly continuous on our sphere. The Dyson-Livesay Theorem states that we can find two points p and q on the sphere S^2 such that $f(p) = f(-p) = f(q) = f(-q)$ and $\measuredangle(p, q) = \alpha$. That means, if we rotated the table in space so that the four table feet fit into the locations $p, -p, q$ and $-q$ and lowered it to the floor it would rest firmly. Therefore, the same result can be accomplished by simply turning the table on the ground, while keeping the intersection of the diagonals on the same vertical line.



(a) (b) (c)

Figure 1: (a) A rectangular table. (b) Spherical motion. (c) A slicing plane.

Scene 2: Our table's wobbling problem was fixed. We got our pizza out of the oven and were about to cut it, when my roommate said: "I want a piece with exactly the same amounts of sausage and pepperoni as your piece". Mathematics got me out of trouble again. The Borsuk-Ulam Theorem assures that there is a straight cut through the pizza which leaves the same amounts of sausage and pepperoni on either side.

This time, imagine a sphere S^2 resting on the table surface. For any point p on the sphere, consider the plane whose normal vector extends from the center of the sphere to p. (See Figure 1(b), 1(c).) Let the function $f(p)$ denote the amount of sausage on the p side of the plane, while $g(p)$ denotes the amount of pepperoni on that side. Clearly, f and g are both continuous. The Borsuk-Ulam Theorem states that there is some point p on the sphere satisfying $f(p) = f(-p)$ and $g(p) = g(-p)$. That is, the plane corresponding to point p simultaneously divides the sausage and the pepperoni into two halves, respectively. The intersection of this plane and the pizza should be the cut we want to make.

Both of these two theorems are special cases of Theorem 1 below, of which we will give a self-contained and elementary proof, following [2]. Along the way, we will also give a proof of another important analytical tool, the Brouwer Fixed-Point Theorem.

Theorem 1. *Let $f, g : S^2 \to \mathbb{R}$ be two continuous real-valued functions and let α be any real number in $[0, \pi]$. Then there are points $p, q \in S^2$ with $\measuredangle(p, q) = \alpha$ such that $f(p) = f(q)$, $g(p) = g(-p)$ and $g(q) = g(-q)$.*

Remark 2.

(a) If we take $f = g$, then $f(p) = f(q) = f(-p) = f(-q)$ with $\measuredangle(p, q) = \alpha$. This special case is known as the Dyson-Livesay Theorem.

(b) If we take $\alpha = \pi$, then $f(p) = f(-p)$ and $g(p) = g(-p)$. This special case is know as the Borsuk-Ulam Theorem.

Proof: Let Y be the boundary of the cube $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] = [-\frac{1}{2}, \frac{1}{2}]^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid -\frac{1}{2} \leqslant x_i \leqslant \frac{1}{2}, i = 1, 2, 3\}$. It is easy to show that Y is topologically equivalent to the sphere S^2 and we can easily find an equivalence between them that does not change corresponding central angles. So we only need to show that the stated theorem is true for functions $f, g : Y \to \mathbb{R}$.

First fix $\epsilon > 0$. Subdivide Y regularly into $6n^2$ small squares (subdivide each face into $n \times n$ subsquares) so that $|g(x) - g(y)| < \epsilon/2$ if x and y lie in the same square, as indicated in Figure 2(a). (This can be done since g is uniformly continuous on its closed and bounded domain Y.)

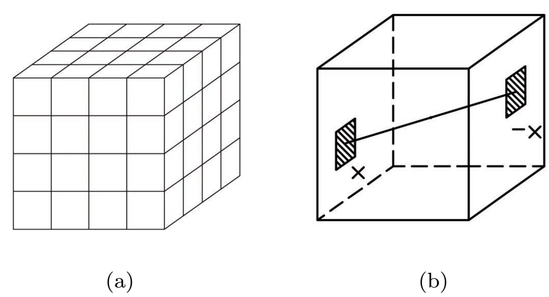


Figure 2: (a) Subdivision into 96 subsquares ($n = 4$). (b) Coloring rule.

For each pair of opposed squares, compare the g values of the midpoints x and $-x$: if $g(x) > g(-x)$, then color the square with center x red and color the one with center $-x$ blue; if $g(x) = g(-x)$ color one red and the other one blue, randomly. (See Figure 2(b).)

We let B denote blue and R denote red. Let F be the color assignment $F : Y' \to \{B, R\}$ where Y' is the set of subsquares we get from the subdivision described above, and under which the antipodal squares have different values. Then for any such mapping F, there exists a simple closed curve along the square grid lines which is symmetric across the center of the cube and which, at all times, borders at least one red and one blue square.

In order to prove the existence of this curve, we imagine the regions labeled "B" and "R" as two different countries and regard each connected region with the same assignment as a state of that country. Then the border lines must be symmetric with respect to the center of the cube. There are two cases:

Case (1): Each of these two countries is itself connected. Then the border between these countries is the desired curve.

Case (2): The states of both countries are interwoven. Then we can find an innermost state (in Figure 3 we consider a state of country "B") and then swap it with the opposed state. We could repeat this process until we are falling into Case (1). Since the resulting border would be part of the original border, before these swaps, we never have to actually swap any regions at all to find the desired curve.

After finding the curve, let L denote its length. We parameterize this curve

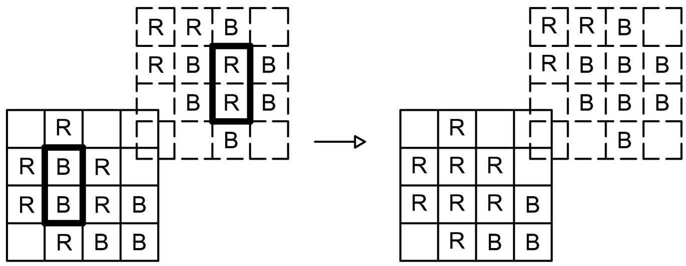


Figure 3: Swapping states

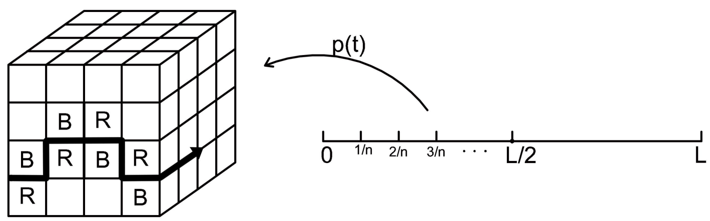


Figure 4: Parameterization for the simple closed curve.

by $p(t)$ using arc-length. (See Figure 4.) Then we define:

$$\varphi(s,t) = \alpha - \measuredangle(p(s), p(s+t)) \tag{1}$$

$$\psi(s,t) = f(p(s)) - f(p(s+t)) \tag{2}$$

Let a and b be numbers in $[0, L]$ satisfying $f(p(a)) = max(f \circ p)$ and $f(p(b)) = min(f \circ p)$. We may assume that $a < b$ (otherwise we can reverse the orientation of p). Then we look at the functional values of φ and ψ in the square $[a, b] \times [0, L/2]$. (See Figure 5.)

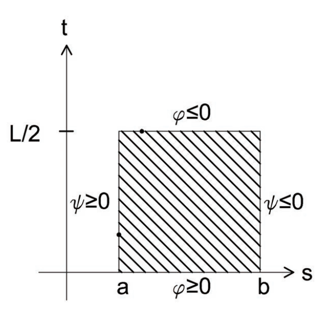


Figure 5: φ and ψ values along the boundary of $[a, b] \times [0, L/2]$.

For any point on the top boundary of the square, say (s_1, t_1), we have $t_1 = L/2$. Then $\measuredangle(p(s_1), p(s_1 + t_1)) = \pi$ because $p(s_1)$ and $p(s_1 + L/2)$ are two opposed points. Since $0 \leqslant \alpha \leqslant \pi$, we get $\varphi(s_1, t_1) \leqslant 0$.

For any point on the left-hand boundary, say (s_2, t_2), we have $s_2 = a$. Since $f(p(a)) = max(f \circ p)$, we get $f(p(a)) \geqslant f(p(a + t_2))$. That is, $\psi(s_2, t_2) \geqslant 0$ for such (s_2, t_2). Similarly, we get $\psi(s, t) \leqslant 0$ for points on the right-hand boundary and $\varphi(s, t) \geqslant 0$ for points on the bottom.

It follows from Corollary 5 below that there is a point $(s_0, t_0) \in [a, b] \times [0, L/2]$ such that $\varphi(s_0, t_0) = 0 = \psi(s_0, t_0)$. Take $p = p(s_0)$ and $q = p(s_0 + t_0)$. Then from (1) and (2), we get

$$\measuredangle(p, q) = \measuredangle(p(s_0), p(s_0 + t_0)) = \alpha - \varphi(s_0, t_0) = \alpha - 0 = \alpha, \text{ and}$$

$$f(p) - f(q) = f(p(s_0)) - f(p(s_0 + t_0)) = \psi(s_0, t_0) = 0.$$

Recall that if x and y lie in the same square then $|g(x) - g(y)| < \epsilon/2$. Let x_1 and x_2 be the midpoints of the squares sharing point p. (See Figure 6.) Then we have

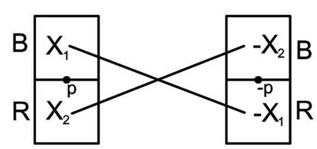


Figure 6: Picture for related points of p.

$$|g(p) - g(x_1)| < \epsilon/2 \Rightarrow -\epsilon/2 < g(p) - g(x_1) < \epsilon/2 \tag{3}$$

$$|g(p) - g(x_2)| < \epsilon/2 \Rightarrow -\epsilon/2 < g(p) - g(x_2) < \epsilon/2 \tag{4}$$

$$|g(-p) - g(-x_1)| < \epsilon/2 \Rightarrow -\epsilon/2 < g(-x_1) - g(-p) < \epsilon/2 \tag{5}$$

$$|g(-p) - g(-x_2)| < \epsilon/2 \Rightarrow -\epsilon/2 < g(-x_2) - g(-p) < \epsilon/2 \tag{6}$$

Combining equations (3), (5) and (4), (6) respectively yields

$$-\epsilon < g(p) - g(-p) - g(x_1) + g(-x_1) < \epsilon \tag{7}$$

$$-\epsilon < g(p) - g(-p) - g(x_2) + g(-x_2) < \epsilon \tag{8}$$

We may have $[x_1, x_2, -x_1, -x_2] = [B, R, R, B]$ or $[x_1, x_2, -x_1, -x_2] = [R, B, B, R]$. We assume the first case is true. Then, according to our labeling rule, $g(x_1) \leqslant g(-x_1)$ so that $\lambda_1 = g(-x_1) - g(x_1) \geqslant 0$. Similarly, $g(-x_2) \leqslant g(x_2)$ so that $\lambda_2 = g(x_2) - g(-x_2) \geqslant 0$. Then equations (7) and (8) become

$$-\epsilon - \lambda_1 < g(p) - g(-p) < \epsilon - \lambda_1, \text{ and}$$

$$-\epsilon + \lambda_2 < g(p) - g(-p) < \epsilon + \lambda_2,$$

so that

$$-\epsilon + \lambda_2 < g(p) - g(-p) < \epsilon - \lambda_1$$

with $\lambda_2, \lambda_1 \geqslant 0$. Then we have $|g(p) - g(-p)| < \epsilon$. This is also true if the second case is chosen. Applying the same method, we get $|g(q) - g(-q)| < \epsilon$.

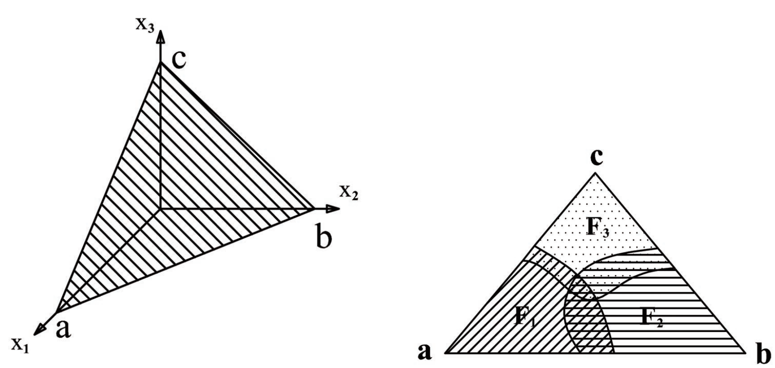
Finally, we consider $\epsilon_n = 1/n$. Then we obtain sequences p_n and q_n as above, which have some subsequences p_{n_k} and q_{n_k} converging to two points in Y, say p and q. Since $\epsilon_{n_k} = 1/n_k$ is going to 0 as p_{n_k} and q_{n_k} are converging to p and q, respectively, we can conclude that $|g(p) - g(-p)| = 0 = |g(q) - g(-q)|$. This completes the proof. □

We are now going to prove the Brouwer Fixed-Point Theorem and its corollary, which we referred to earlier.

Theorem 3 (Brouwer Fixed-Point Theorem)**.** *For every continuous function* $f : [0,1]^n \to [0,1]^n$, *there is at least one* $x \in [0,1]^n$ *such that* $f(x) = x$.

Proof: When $n = 1$, we have $f : [0,1] \to [0,1]$. We define $g(x) = f(x) - 1$, whose domain is $[0,1]$. Since $0 \leqslant f(x) \leqslant 1$, we have $g(0) = f(0) - 0 \geqslant 0$ and $g(1) = f(1) - 1 \leqslant 0$. If in either of the two inequalities the equality holds, that is, if either $f(0) = 0$ or $f(1) = 1$, then either $x = 0$ or $x = 1$ is a fixed point, respectively. So, suppose neither equality holds. Then $g(0) = f(0) - 0 > 0$ and $g(1) = f(1) - 1 < 0$. Since $g(x)$ is continuous, there is some x in $(0,1)$ such that $g(x) = 0$. That is, $f(x) = x$.

For the case $n = 2$, we replace the square $[0,1]^2$ by the topologically equivalent triangle $X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1, x_1 \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0\}$ with vertices $a = (1,0,0)$, $b = (0,1,0)$ and $c = (0,0,1)$, and consider a continuous function $f : X \to X$. (See Figure 7(a).)



(a) (b)

Figure 7: (a) X in $\mathbb{R}^3$. (b) Three regions in X.

Let $\alpha_i : \mathbb{R}^3 \to \mathbb{R}$ denote the projection onto the i-th coordinate axis given by $\alpha_i(x_1, x_2, x_3) = x_i$. Then for $x = (x_1, x_2, x_3) = (\alpha_1(x),\ \alpha_2(x),\ \alpha_3(x)) \in X$ we have $\alpha_1(x) + \ \alpha_2(x) + \ \alpha_3(x) = 1$ and $\alpha_1(f(x)) + \alpha_2(f(x)) + \alpha_3(f(x)) = 1$. We define the following three regions in X (see Figure 7(b)):

$$F_1 = \{x \in X | \alpha_1(f(x)) \leqslant \alpha_1(x)\}$$

$$F_2 = \{x \in X | \alpha_2(f(x)) \leqslant \alpha_2(x)\}$$
$$F_3 = \{x \in X | \alpha_3(f(x)) \leqslant \alpha_3(x)\}$$

It is easy to show that each of F_1, F_2 and F_3 is a closed subset of X. Since $\alpha_1(f(a)) \leqslant 1 = \alpha_1(a)$, we have $a \in F_1$. Similarly $b \in F_2$ and $c \in F_3$.

Note that $X = F_1 \cup F_2 \cup F_3$. (If there were some point $d \in X \setminus (F_1 \cup F_2 \cup F_3)$, then $1 = \alpha_1(f(d)) + \alpha_2(f(d)) + \alpha_3(f(d)) > \alpha_1(d) + \alpha_2(d) + \alpha_3(d) = 1$.)

For every point $x \in F_1 \cap F_2 \cap F_3$, we have $f(x) = x$, since for such x equality must hold in each of the three inequalities defining the sets F_1, F_2 and F_3. So, we only need to show that $F_1 \cap F_2 \cap F_3 \neq \emptyset$.

Suppose, to the contrary, that $F_1 \cap F_2 \cap F_3 = \emptyset$. Then the three open subsets $X \setminus F_1, X \setminus F_2$ and $X \setminus F_3$ of X cover $X = (X \setminus F_1) \cup (X \setminus F_2) \cup (X \setminus F_3)$. Choose a number $\lambda > 0$, sufficiently small, so that every subset of X of diameter less than λ lies in at least one of the sets $X \setminus F_1, X \setminus F_2$ or $X \setminus F_3$. (This is possible by the Lebesgue Number Lemma [3].)

We assign to the three vertices a, b and c, the numbers 1, 2 and 3, respectively. Then we subdivide the triangle X into six smaller triangles, using its three medians. (Recall that the three medians meet at the centroid of the triangle.) Since $X = F_1 \cup F_2 \cup F_3$, we can assign to each new vertex y a number $i = i(y) \in \{1, 2, 3\}$ such that $y \in F(i)$. (See Figure 8.) We further

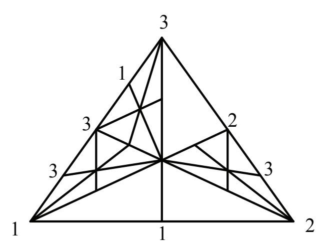


Figure 8: One assignment for crossing points.

subdivide the smaller triangles in the same fashion and assign numbers to the resulting new vertices according to the same rule. Here we can arrange that, given any one of the three sides of the original triangle X, the vertices on that side have only two different kinds of labels: each vertex y on the side $\overline{ab}$ of X, for example, can be given a label from the set $\{1, 2\}$; for otherwise $y \notin F_1 \cup F_2$, implying that $1 = \alpha_1(f(y)) + \alpha_2(f(y)) + \alpha_3(f(y)) \geqslant \alpha_1(f(y)) + \alpha_2(f(y)) > \alpha_1(y) + \alpha_2(y) = \alpha_1(y) + \alpha_2(y) + \alpha_3(y) = 1$.

We subdivide X sufficiently fine, so that every subtriangle has diameter less than λ. By Lemma 4 below, there is at least one subtriangle A whose three vertices have all three labels 1, 2 and 3, in some order. But then $A \not\subseteq X/F_i$ for all $i \in \{1, 2, 3\}$, according to our labeling rule. This contradicts our choice of λ and proves the theorem for $n = 2$. The proof for $n > 2$ is entirely analogous. $\square$

The set of all points $(x_1, x_2, \ldots, x_{n+1})$ in $\mathbb{R}^{n+1}$ with $x_1 + x_2 + \cdots + x_{n+1} = 1$ and $x_i \geqslant 0$ is called an *standard n-dimensional simplex*. If, in addition, we set some selection (at least one, but not all) of the coordinates $x_i = 0$, we obtain a so-called *face* of this simplex. A face of a simplex is also simplex, but of lower dimension. The intercepts with the coordinate axes are called the *vertices* of the simplex. Any image of a standard simplex under an invertible linear transformation is also called a simplex.

Lemma 4 (Sperner's Lemma)**.** *Suppose an n-dimensional simplex Δ has been triangulated into subsimplices and that each vertex of the triangulation has been given a "color" from the set $\{1, 2, \dots, n+1\}$. Suppose, further, that each of the $n+1$ vertices of Δ has a different color and that every vertex v of the triangulation which lies on a face S of Δ (of any dimension) has a color that matches the color of some vertex w of S. Then there is an n-dimensional subsimplex in this triangulation all whose $n+1$ vertices have different colors.*

Figure 9: Sperner's Lemma in dimension $n = 1$.

Proof: By induction on n, one proves that the number of fully colored subsimplices is odd. We will prove this for $n = 1$ and illustrate the general induction step with the implication from $n = 1$ to $n = 2$.

$n = 1$: Consider a fully colored 1-dimensional simplex, that is, an interval whose endpoints are colored 1 and 2, respectively. Then for any colored subdivision, the number of fully colored subsimplices must be odd, since each such subinterval represents a switch from one color to the other color. (See Figure 9.)

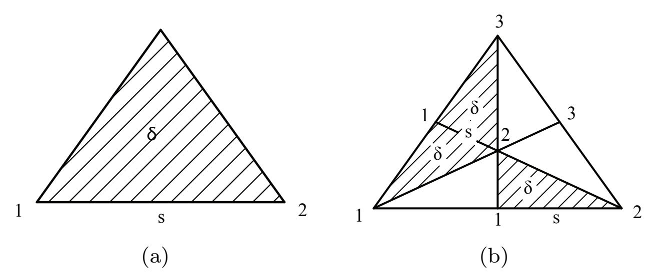


Figure 10: (a) One pair (s, δ). (b) Two situations of Method 1.

$n = 2$: Consider a fully colored 2-dimensional simplex Δ, that is, a triangle whose three vertices are colored 1, 2 and 3, respectively. Let a subdivision of Δ be given, colored according to the statement of the lemma. We will count, in two different ways, all pairs (s, δ) where s is a side of a subdivision triangle δ and the endpoints of s are colored 1 and 2, respectively. (See Figure 10(a).)

Method 1: Fix s and consider Figure 10(b). Either the side s is interior to Δ or it is at the bottom, but it is never contained in any of the other two boundary sides. If there are n_1 such s's in the interior, then there are $2n_1$

corresponding pairs (s, δ). If there are n_2 such s's at the bottom, then there are n_2 corresponding pairs (s, δ). So the total number of pairs (s, δ) is $2n_1 + n_2$. The bottom side is a 1-dimensional simplex and we have proved above that n_2 is odd, so that $2n_1 + n_2$ is odd. (In dimension n, the corresponding n_2 is odd by induction hypothesis, since the "bottom side" is an $(n-1)$-dimensional simplex satisfying the assumptions of the lemma.)

Method 2: Now fix δ. Then the subtriangle δ is either fully colored or it is not. If the number of fully colored δ's is m_1, then there are m_1 corresponding pairs (s, δ). (See Figure 11(a).) If the number of non-fully colored δ's is m_2, then there are $2m_2$ corresponding pairs (s, δ). (See Figure 11(b).)

So, the total number of pairs (s, δ) is $m_1 + 2m_2 = 2n_1 + n_2$ which is odd. Since $2m_2$ is even, we see that m_1 is odd. That is, the number of fully colored subtriangles is odd. □

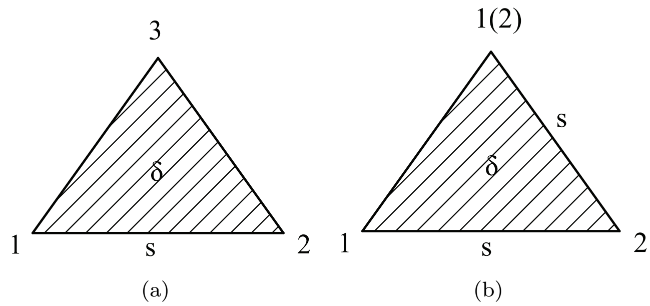


Figure 11: (a) Full subtriangle. (b) Non-full subtriangle.

Corollary 5. *Let $\varphi, \psi : [-1,1]^2 \to \mathbb{R}$ be two continuous functions such that*

$$\varphi(s,-1) \geqslant 0,\ \varphi(s,1) \leqslant 0,\ \psi(-1,t) \geqslant 0,\ \psi(1,t) \leqslant 0$$

for all s and t. Then there exists some point (s_0, t_0) in $[-1,1]^2$ such that $\varphi(s_0,t_0) = 0$ and $\psi(s_0,t_0) = 0$.

Proof: Suppose, to the contrary, that there is no point (s,t) with $\psi(s,t) = 0$ and $\varphi(s,t) = 0$. Then the continuous function $F(s,t) = (\psi(s,t), \varphi(s,t))$ is never equal to $(0,0)$. Define a continuous function $G : [-1,1]^2 \to [-1,1]^2$ as follows. Draw a ray from the origin to the point $F(s,t)$. Let $G(s,t)$ be the point where this ray intersects the boundary of the square $[-1,1]^2$. (See Figure 12.)

Since the interior points are all mapped to the boundary, G has no fixed point in the interior. For points (s,t) on the top boundary, we have $\varphi \leqslant 0$, thus $F(s,t) = (\psi(s,t), \varphi(s,t))$ is located in the lower half plane. The corresponding $G(s,t)$ is therefore not on the top boundary. So, no point on the top boundary is a fixed point for G. Similarly, it can be shown that no point on the other three boundaries is a fixed point for G. But this is a contradiction to the Brouwer Fixed-Point Theorem. □

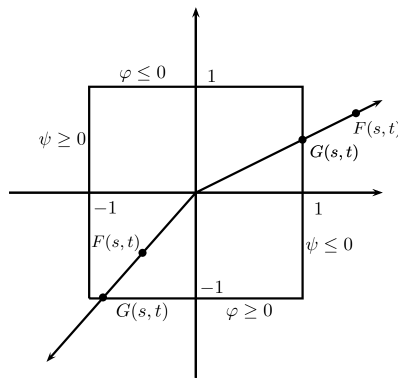


Figure 12: Functions F and G.

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