Student-Faculty Seminar

Wavelets on $\mathbb{Z}_N$

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Introduction

A function, or a signal as electrical engineers like to call it, can be decomposed as a sum, possibly infinite, of sines and cosines, called its Fourier series expansion. The coefficients in this decomposition represent the various frequencies that are present in the signal. However the Fourier expansion fails to give information on what part of the function the frequencies come from. Thus if a signal is a function of time, a Fourier analysis of this signal does not tell us when various of its frequencies occurred. This poses a serious drawback in many applications. A lot of effort has gone into a search for alternative mechanisms, mechanisms whereby a signal can be decomposed into constituents that bear information both on its frequencies and where these frequencies occur in the signal. Luckily, the last 20 years has seen considerable progress due to the discovery of various functions, called wavelets, that roughly speaking replace the cosine and sine functions in the Fourier expansion scheme. Our objective here is to provide the reader with the basics of wavelet construction for the analysis of periodic functions on $\mathbb{Z}$. In spite of the drawback that Fourier analysis has, we do not want to give the impression that it is an outdated method. It has served mathematics and the applied sciences extremely well for over 4 centuries and it remains indispensable in various modern applications. In fact the construction of wavelets depends heavily on Fourier analysis and the practicality and usefulness of wavelets derives from features that are built into the Discrete Fourier Transform, a concept we will look into first. I first came across wavelets in the Student-Faculty Seminar. This is a seminar on wavelets that was run by the department of Mathematical Sciences at Ball State during the 2003/2004 academic year. For this seminar, I was required to have had Linear Algebra experience. Recommendations included Complex Analysis and minor information from Real Analysis. Overall, seminars like these are very undergraduate friendly, with the textbook going through many linear algebra, real analysis, and complex ideas for the theorems needed. If ever the textbook is not helpful, the professors are more than willing to help out with any questions you have.

The Discrete Fourier Transform

Let N be an arbitrary but fixed positive integer. Complex-valued functions on the set

$$\mathbb{Z}_N := \{0, 1, \ldots, N-1\}$$

will be denoted by $\ell^2(\mathbb{Z}_N)$. It will be convenient to think of such functions as being extended to the whole set of integers $\mathbb{Z}$ as N-periodic complex valued functions. Thus if $z \in \ell^2(\mathbb{Z}_N)$, then $z(n + kN) = z(n)$ for all $n \in \mathbb{Z}_N$ and all $k \in \mathbb{Z}$.

It is easy to see that $\ell^2(\mathbb{Z}_N)$ is a vector space over the complex field with respect to the usual addition of functions and multiplication of functions by a scalar. In fact the following inner product turns $\ell^2(\mathbb{Z}_N)$ into an inner product space.

$$\langle z, w \rangle = \sum_{k=0}^{N-1} z(k)\overline{w(k)}, \quad z, w \in \mathbb{Z}_N.$$

We define $F_n \in \ell^2(\mathbb{Z}_N)$ as follows:

$$F_n(k) = \frac{1}{N} e^{2\pi ikn/N}, \quad k \in \mathbb{Z}_N.$$

Then $\mathcal{F} := \{F_0, F_1, \ldots, F_{N-1}\}$ forms an orthogonal basis of $\ell^2(\mathbb{Z}_N)$ called the Fourier basis. Note that $\|F_n\| = 1/\sqrt{N}$. The coordinate vector of $z \in \mathbb{Z}_N$ with respect to the ordered basis $\mathcal{F}$ will be denoted by $\hat{z}$. Thus,

$$z = \hat{z}(0)F_0 + \hat{z}(1)F_1 + \ldots + \hat{z}(N-1)F_{N-1}.$$

Therefore

$$\hat{z}(n) = N\langle z, F_n \rangle = N \sum_{k=0}^{N-1} z(k)\overline{F_n(k)} = \sum_{k=0}^{N-1} z(k) e^{-2\pi ikn/N}.$$

Definition 1. Given $z \in \ell^2(\mathbb{Z}_N)$, we let

$$\hat{z}(n) = \sum_{k=0}^{N-1} z(k) e^{-2\pi ikn/N}.$$

The mapping

$$\hat{}: \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N),$$

which takes z to $\hat{z}$, is called the **discrete Fourier transform** (or DFT, for short). It is easily checked that $\hat{z}$ is N-periodic.

The following theorem is useful.

Theorem 2 (Fourier Inversion Formula).

$$z(n) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{z}(k) e^{2\pi ikn/N}, \quad n \in \mathbb{Z}_N, \quad z \in \ell^2(\mathbb{Z}_N).$$

The above inversion formula shows that the DFT is a one-to-one linear transformation on the inner product space $\ell^2(\mathbb{Z}_N)$. The inverse $\check{}: \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)$ of $\hat{}: \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)$, called the **inverse discrete Fourier transform** or (IDFT for short), is the linear transformation defined explicitly as

$$\check{w}(n) = \frac{1}{N} \sum_{k=0}^{N-1} w(k) e^{2\pi ikn/N}, \quad n \in \mathbb{Z}_N, \quad w \in \ell^2(\mathbb{Z}_N).$$

Definition 3.

1. For any $w \in \ell^2(\mathbb{Z}_N)$ we define the **conjugate reflection** $\tilde{w}$ as the element in $\ell^2(\mathbb{Z}_N)$ given by
$$\tilde{w}(n) = \overline{w(-n)}, \quad n \in \mathbb{Z}.$$
2. For $z, w \in \ell^2(\mathbb{Z}_N)$, the **convolution** of z and w, denoted by $z * w$ is an element in $\ell^2(\mathbb{Z}_N)$ defined as
$$z * w(n) = \sum_{k=0}^{N-1} z(n-k)w(k), \quad n \in \mathbb{Z}.$$

One can easily check that the conjugate reflection and convolution operations transform as follows under the DFT:

$$\hat{\tilde{w}}(n) = \overline{\hat{w}(n)} \text{ and } \widehat{z * w}(n) = \hat{z}(n)\hat{w}(n),$$

where the latter expression is componentwise multiplication. We now define another useful linear transformation on $\ell^2(\mathbb{Z}_N)$.

Definition 4. For each $k \in \mathbb{Z}$ we define $R_k : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)$ as

$$R_k z(n) = z(n-k), \quad n \in \mathbb{Z}, \quad z \in \ell^2(\mathbb{Z}_N).$$

R_k is called the **translation by k operator**.

The following lemma will be useful.

Lemma 5. *For all $k, n \in \mathbb{Z}$ we have*

$$(i) \quad \widehat{R_k z}\,(n) = e^{-2\pi ink/N}\hat{z}(n), \qquad (ii) \quad \langle z, R_{2k} u \rangle = z * \tilde{w}(k).$$

The Fast Fourier Transform

In carrying out operations, things can sometimes be arranged so that certain features of the operation can be exploited cleverly in such a way that would speed up the operation. In 1965, Cooley and Tukey took advantage of a useful feature of the DFT that helped cut down on the time needed to compute the Fourier transform. Their ideas essentially revolutionized the science of computing.

Here we touch on this aspect of Fourier transform computation. It is called the Fast Fourier Transform (FFT).We start by considering the simplest case when N is even to highlight the basic idea behind the FFT.

Lemma 6. *Suppose $N = 2M$. Given $z \in \ell^2(\mathbb{Z}_N)$, define $u, v \in \ell^2(\mathbb{Z}_M)$ by*

$$u(k) = z(2k), \qquad \textit{and} \quad v(k) = z(2k+1), \qquad k \in \mathbb{Z}.$$

Then

$$\begin{aligned} \hat{z}(n) &= \hat{u}(n) + e^{-2\pi n/N}\hat{v}(n), \qquad n \in \mathbb{Z}_M, \quad \textit{and} \\ \hat{z}(n+M) &= \hat{u}(n) - e^{-2\pi n/N}\hat{v}(n), \qquad n \in \mathbb{Z}_M. \end{aligned}$$

Notice that in the above lemma the same values of $\hat{u}(n)$ and $\hat{v}(n)$ are used in computing both $\hat{z}(n)$ and $\hat{z}(n+M)$ for all $n \in \mathbb{Z}_M$. First we compute $\hat{u}$ and $\hat{v}$ at $n = 0, \ldots, M-1$. This requires M^2 multiplications each. Then we compute $e^{-2\pi i n/N}\hat{v}(n)$ for $n = 0, \ldots, M$ which requires an additional M multiplications. Thus, at most $2M^2+M = N(N+1)/2$ multiplications are required to compute $\hat{z}$ for $z \in \ell^2(\mathbb{Z}_N)$, by this scheme. This is essentially $N^2/2$ for large N, thus cutting down the computation of $\hat{z}$ by half of what would normally be needed using direct methods. This procedure can be iterated if N is divisible by higher powers of 2. The best situation for this scheme occurs when N is a power of two. In this case, an iteration of the above argument leads to the following lemma.

Lemma 7. *Suppose N is a power of 2. Then the DFT of an element of $\ell^2(\mathbb{Z}_N)$ can be computed with no more than $\frac{1}{2}N\log_2 N$ complex multiplications.*

The FFT can also be used to compute the IDFT fast (and hence convolutions as well), since

$$\check{w}(n) = \frac{1}{N}\hat{w}(N-n), \qquad \text{and} \qquad z * w = (\hat{z}\hat{w})\check{}.$$

Wavelets in $\mathbb{Z}_N$ - The First Stage

The idea here is to construct an orthonormal basis $\{\psi_k\}_{k=0}^{N-1}$ of $\ell^2(\mathbb{Z}_N)$ such that each ψ_k is localized in space as well as frequency. That is, we would want each ψ_k and its DFT $\hat{\psi}_k$ to contain as many zeros as possible. This would allow us to focus on local spacial or frequency features of a signal when it is expressed in such a basis.

The FFT can be used efficiently to compute coefficients of signals in such basis. (See Part (2) of Lemma 5.)

Lemma 8. *Let $N = 2M$, and $w \in \ell^2(\mathbb{Z}_N)$. The set $\{R_{2k}w\}_{k=0}^{M-1}$ is an orthonormal set if and only if*

$$|\hat{w}(n)|^2 + |\hat{w}(n+M)|^2 = 2 \text{ for all } n = 0, \ldots, M-1. \quad (1)$$

We remark that if $u \in \ell^2(\mathbb{Z}_N)$ satisfies (1) above, and if we define $v \in \ell^2(\mathbb{Z}_N)$ to be

$$v(k) = (-1)^k u(1-k), \quad k \in \mathbb{Z}_N, \quad (2)$$

then one can show that the following relations hold for all $n \in \mathbb{Z}_N$.

$$|\hat{u}(n)|^2 + |\hat{u}(n+M)|^2 = 2, \quad |\hat{v}(n)|^2 + |\hat{v}(n+M)|^2 = 2, \quad (3)$$
$$\hat{u}(n)\overline{\hat{v}(n)} + \hat{u}(n+M)\overline{\hat{v}(n+M)} = 0. \quad (4)$$

The above conditions force the set

$$\{u, R_2u, \ldots, R_{M-1}u\} \cup \{v, R_2v, \ldots, R_{M-1}v\}$$

to be an orthonormal basis of $\ell^2(\mathbb{Z}_N)$.

This lemma illustrates the fact that orthonormal bases depend on our DFT.

Definition 9. Let $N = 2M$ for some positive integer M. An orthonormal basis of $\ell^2(\mathbb{Z}_N)$ of the form

$$\{R_{2k}u,\ R_{2k}v\}_{k=0}^{M-1}$$

for some $u, v \in \ell^2(\mathbb{Z}_N)$ is called a **first–stage wavelet basis** for $\ell^2(\mathbb{Z}_N)$.

It is important to note that given $z \in \ell^2(\mathbb{Z}_N)$, the coefficients $\langle z, R_{2k}u\rangle = z * \tilde{u}(k)$ can be computed via the FFT scheme, since $z * w = (\hat{z}\hat{w})\check{}$. The same remark applies to the coefficients $\langle z, R_{2k}v\rangle$.

Theorem 10. *Elements $u, v \in \ell^2(\mathbb{Z}_N)$ generate a first–stage wavelet if and only if their system matrix*

$$A(n) = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{u}(n) & \hat{v}(n) \\ \hat{u}(n+M) & \hat{v}(n+M) \end{bmatrix}$$

is a unitary matrix for all $n \in \mathbb{Z}$.

In more explicit terms, the above theorem states that u, v and their translates form a basis of $\ell^2(\mathbb{Z}_N)$ if and only if

$$|\hat{u}(n)|^2 + |\hat{u}(n+M)|^2 = 2, \quad |\hat{v}(n)|^2 + |\hat{v}(n+M)|^2 = 2,$$

$$\hat{u}(n)\overline{\hat{v}(n)} + \hat{u}(n+M)\overline{\hat{v}(n+M)} = 0.$$

Now that we have established the u and v wavelet basis, we can now get into how we compress. We now define a helpful operation.

Definition 11. Suppose $N = 2M$ for some positive integer M. Define $U : \ell^2(\mathbb{Z}_M) \to \ell^2(\mathbb{Z}_N)$ by setting

$$U(z)(n) = \begin{cases} z(n/2) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

for $z \in \ell^2(\mathbb{Z}_M)$ and $n = 0, 1, ..., M-1$. We call the operator U the **upsampling operator**. We denote U by $\uparrow 2$.

We can now use all these definitions, lemmas, and theorems to define a wavelet basis and figure out a wavelet filter sequence with repeated filters. This scheme will show how we can reconstruct a signal using wavelet compression.

Wavelets in $\mathbb{Z}_N$ - The Iteration Stage

Suppose we would like to determine the presence of an object through a radar or sonar search. In analyzing the signal from such search our initial concern is to get enough information from the signal to decide whether the object of interest (e.g., oil deposit) is present. At this stage, details are less relevant. Another scenario would be the transmission of data via a data line. Compression of the data to an acceptable level before transmission could save a lot of money or time (or both). Therefore the signals that come from such situations could be analyzed using a basis that gives less detailed information about the signal. Thus to obtain only crude information about the signal, we need to obtain wavelet basis at scales higher than 2, in the sense to be explained below. We suppose that N is divisible by 2^p for some fixed $p > 1$. We start out with $u = u_1$ and $v = v_1$ in $\ell^2(\mathbb{Z}_N)$ whose system matrix is unitary. We obtain a p^{th}-stage **wavelet filter sequence** $u_\ell, v_\ell \in \ell^2(\mathbb{Z}_{N/2^{\ell-1}}), \quad \ell = 1, 2, \ldots, p$, as follows. Let

$$u_\ell(n) = \sum_{k=0}^{2^{\ell-1}-1} u_1\left(n + \frac{kN}{2^{\ell-1}}\right) \quad \text{and} \quad v_\ell(n) = \sum_{k=0}^{2^{\ell-1}-1} v_1\left(n + \frac{kN}{2^{\ell-1}}\right).$$

We then define from u_ℓ and v_ℓ, elements whose translates will provides us with particular basis elements. They are defined as follows:

$$f_1 = v_1 \quad \text{and} \quad g_1 = u_1,$$

$$f_\ell = g_{\ell-1} * U^{\ell-1}(v_\ell) \quad \text{and} \quad g_\ell = g_{\ell-1} * U^{\ell-1}(u_\ell).$$

Now we let

$$\psi_{-j,k} = R_{2^j k}\, f_j \quad \text{and} \quad \phi_{-j,k} = R_{2^j k}\, g_j$$

for $j = 1, \ldots, p$ and $k = 0, \ldots, (N/2^j) - 1$.
Then,

$$\{\psi_{-1,k}\}_{k=0}^{(N/2)-1} \cup \{\psi_{-2,k}\}_{k=0}^{(N/4)-1} \cup \cdots \cup \{\psi_{-p,k}\}_{k=0}^{(N/2^p)-1} \cup \{\phi_{-p,k}\}_{k=0}^{(N/2^p)-1}$$

is an orthonormal basis for $\ell^2(\mathbb{Z}_N)$, called a p^{th}**-stage wavelet basis**.

We observe that for any $m = 1, \dots, p$,

$$\{\psi_{-1,k}\}_{k=0}^{(N/2)-1} \cup \{\psi_{-2,k}\}_{k=0}^{(N/4)-1} \cup \dots \cup \{\psi_{-m,k}\}_{k=0}^{(N/2^m)-1} \cup \{\phi_{-m,k}\}_{k=0}^{(N/2^m)-1}$$

is an m^{th}-stage wavelet basis. We now introduce a sequence of subspaces of $\ell^2(\mathbb{Z}_N)$ as follows:

$$V_{-j} := \text{Span}\{\phi_{-j,k}\}_{k=0}^{(N/2^j)-1}, \qquad W_{-j} := \text{Span}\{\psi_{-j,k}\}_{k=0}^{(N/2^j)-1}.$$

One can show that for all $j = 1, \dots, p$, the spaces V_{-j} and W_{-j} are subspaces of V_{-j+1} and that

$$V_{-j} \oplus W_{-j} = V_{-j+1}, \tag{5}$$

where we let V_0 stand for $\ell^2(\mathbb{Z}_N)$.

We have found our orthonormal basis and we can send the signal through to compress it. We have to use our ψ's and ϕ's to reproduce this signal. We want to have a raw estimation of z, so we project z onto the basis that we found. We will call this projection

$$P_{-j}(z) = \sum_{k=0}^{(N/2^j)-1} \langle z, \phi_{-j,k} \rangle \phi_{-j,k}.$$

Similarly, we define

$$Q_{-j}(z) = \sum_{k=0}^{(N/2^j)-1} \langle z, \psi_{-j,k} \rangle \psi_{-j,k}.$$

From the relation in (5) we immediately see that for any $z \in \ell^2(\mathbb{Z}_N)$,

$$P_{-j+1}(z) = P_{-j}(z) + Q_{-j}(z), \quad j = 1, 2, 3, \dots, p.$$

Here we have used P_0 for the identity operator $P_0(z) = z, \quad z \in \ell^2(\mathbb{Z}_N)$. These equations can be interpreted as follows. Given a signal z, the projection $P_{-j+1}(z)$ gives its $(-j+1)^{th}$ approximation (a compression of the signal), and $Q_{-j}(z)$ contains the "details at level $(-j+1)$" that one could use to obtain a better approximation $P_{-j+1}(z)$ from its $(-j)^{th}$ level approximation $P_{-j}(z)$.

Wavelet Construction - An Example

To give the reader a flavor of the general construction of wavelets discussed in the previous section, we look at a particular example. The example we give below belongs to a class of wavelets first constructed by Ingrid Daubechies. We will assume that $N > 64$. We want to construct $u \in \ell^2(\mathbb{Z}_N)$ such that its system matrix is unitary and u has only four nonzero components. For this we let

$$b(n) = \cos^6\left(\frac{\pi}{N}n\right) + 3\cos^4\left(\frac{\pi}{N}n\right)\sin^2\left(\frac{\pi}{N}n\right).$$

Then note that

$$b(n+8) = 3\cos^2\left(\frac{\pi}{N}n\right)\sin^4\left(\frac{\pi}{N}n\right) + \sin^6\left(\frac{\pi}{N}n\right).$$

Moreover

$$b(n) + b(n+8) = \left(\cos^2\left(\frac{\pi}{N}n\right) + \sin^2\left(\frac{\pi}{N}n\right)\right)^3 = 1. \tag{6}$$

If we now define $u \in \ell^2(\mathbb{Z}_N)$ such that

$$|\hat{u}(n)|^2 = 2b(n), \tag{7}$$

then Equation (6) leads to

$$|\hat{u}(n)|^2 + |\hat{u}(n+8)|^2 = 1, \quad n = 0, 1, 2, 3.$$

While there are many ways to pick $u \in \ell^2(\mathbb{Z}_N)$ such that (7) holds, to ensure that such a u has only 4 nonzero components we proceed as follows. We rewrite $b(n)$ as

$$b(n) = \cos^2\left(\frac{\pi}{N}n\right)\left[\cos^4\left(\frac{\pi}{N}n\right) + 3\cos^2\left(\frac{\pi}{N}n\right)\sin^2\left(\frac{\pi}{N}n\right)\right].$$

We now let

$$\begin{aligned}
\hat{u}(n) &= \sqrt{2}e^{-3\pi in/N}\left[\cos\left(\frac{\pi}{N}n\right) - i\sqrt{3}\cos\left(\frac{\pi}{N}n\right)\sin\left(\frac{\pi}{N}n\right)\right] \\
&= \frac{\sqrt{2}}{2}e^{-3\pi n/N}\left(e^{i\pi n/N} + e^{-i\pi n/N}\right)\left[\frac{1}{2}\left(1 + \cos\left(\frac{2\pi n}{N}\right)\right) + i\frac{\sqrt{3}}{2}\sin\left(\frac{2\pi n}{N}\right)\right] \\
&= \left(e^{-2\pi in/N} + e^{-4\pi in/N}\right) \times \\
&\times \left[\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{8}\left(e^{2\pi in/N} + e^{-2\pi in/N}\right) + \frac{\sqrt{6}}{8}\left(e^{2\pi in/N} - e^{-2\pi in/N}\right)\right].
\end{aligned}$$

Since

$$\hat{u}(n) = \sum_{k=0}^{3} u(k)e^{-2\pi ikn/N},$$

direct computation leads to

$$(u(0),\ u(1),\ u(2),\ u(3)) = \frac{\sqrt{2}}{8}(1+\sqrt{3},\ 3+\sqrt{3},\ 3-\sqrt{3},\ 1-\sqrt{3})$$

and $u(n) = 0$ for $n = 4, \ldots, N-1$.

Starting with $u \in \ell^2(\mathbb{Z}_N)$ as above and defining v according to (2) the associated a first stage wavelet basis of $\ell^2(\mathbb{Z}_N)$ is provided by

$$\begin{aligned}
u_1 &= \frac{\sqrt{2}}{8}(1+\sqrt{3},\ 3+\sqrt{3},\ 3-\sqrt{3},\ 1-\sqrt{3},\ 0,\ 0,\ \ldots,\ 0) \\
v_1 &= \frac{\sqrt{2}}{8}(3+\sqrt{3},\ -1-\sqrt{3},\ 0,\ \ldots,\ 0,\ 1-\sqrt{3},\ -3+\sqrt{3}),
\end{aligned}$$

and their even translates.

The recipe described in Section 5 then leads to the construction of the desired p-th stage Daubechies' wavelet basis of $\ell^2(\mathbb{Z}_N)$ for any p for which 2^p divides N and $N/2^p > 4$. We refer the reader to [1] and [2] for more details on wavelets.

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References

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