Articles

A coarse invariant

A. Fox, B. LaBuz, and R. Laskowsky

A. Fox lives in Madison, Wisconsin, where he works as a software developer for Epic, an industry leader in Electronic Health Records. In his spare time he enjoys cooking and tabletop games. Addison has a B.S. in Mathematics from Saint Francis University.

B. LaBuz is an Assistant Professor of Mathematics at Saint Francis University. He has been there for four years and has taught Calculus III, Set Theory, Abstract Algebra, and Foundations of Geometry among other courses.

R. Laskowsky is a senior Mathematics and Computer Science double major student at Saint Francis University. He intends to attend graduate school or work as a computer programmer after graduation.

Abstract This note extends the invariant of metric spaces under bornologous equivalences defined in [1] to the coarse category.

Introduction

Large scale geometry is the study of the large scale behavior of metric spaces– the behavior at infinity. We consider two metric spaces to be equivalent if they have the same behavior at infinity. The standard example of two spaces that are large scale equivalent are the integers $\mathbb{Z}$ and the real numbers $\mathbb{R}$. We can see that these spaces are equivalent by "zooming out." At a certain time the space between the integers becomes indistinguishable and the integers look exactly like the real numbers.

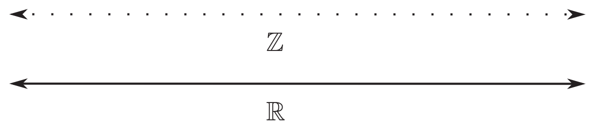

Figure 1: The integers and real numbers are large scale equivalent.

We work in the coarse category defined by Roe [2]. A coarse function $f : X \to Y$ between metric spaces is a function that is bornologous and proper. f is bornologous if for each $N > 0$ there is an $M > 0$ such that if $d(x, y) \le N$, $d(f(x), f(y)) \le M$. In this setting we call f proper if inverse images of bounded sets are bounded.

Notice bornology is dual to continuity. Thus bornology is a fundamental concept of coarse (or large scale) geometry just as continuity is a fundamental concept of topology (small scale geometry). We are studying the large scale behavior of functions and large scale properties of spaces.

Two metric spaces X and Y are coarsely equivalent if there are coarse functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is close to id_X and $f \circ g$ is close to id_Y. Two functions f_1 and f_2 are close if $d(f_1(x), f_2(x))$ is uniformly bounded. A standard reference for the preceding concepts and coarse geometry in general is [2].

In [1] an invariant in the bornologous category is constructed. This note extends the construction in [1] to the coarse category. Bornologous equivalence is more strict than coarse equivalence. For bornologous equivalence $f \circ g$ and $g \circ f$ are required to be the identity. Coarse equivalence can be viewed as being in the category where, instead of considering functions, one considers equivalence classes of functions. Two functions are equivalent if they are close.

The standard example of two coarsely equivalent spaces is $\mathbb{R}$ and $\mathbb{Z}$ (see Example 5). Of course these spaces cannot be bornologously equivalent because they do not have the same cardinality. We can explain interest in the coarse category as opposed to the bornologous category as follows. Since we are interested in large scale behavior, we should ignore all small scale behavior including cardinality. We should not care whether the number of points in a neighborhood is finite or infinite.

Previous construction

We recall the construction from [1]. Fix a basepoint $x_0 \in X$. Given $N > 0$, an N-sequence in X based at x_0 is an infinite list $x_0, x_1, \dots$ of points in X with $d(x_i, x_{i+1}) \le N$ for each $i \ge 0$. Since we are interested in the large scale structure of X, we are only interested in sequences that go to infinity. An N-sequence $x_0, x_1, \dots$ goes to infinity if $d(x_0, x_i) \to \infty$. Let $\mathrm{S}_N(X, x_0)$ be the set of all N-sequences in X based at x_0 that go to infinity.

We call two sequences $s, t \in \mathrm{S}_N(X, x_0)$ equivalent if there is a finite list $s_0, \dots, s_n \in \mathrm{S}_N(X, x_0)$ with $s_0 = s$, $s_n = t$, and for each $i \ge 0$, s_{i+1} is either a subsequence of s_i or s_i is a subsequence of s_{i+1}. If s_i is a subsequence of s_{i+1}

we say s_{i+1} is a supersequence of s_i. Let $[s]_N$ denote the equivalence class of s in $\mathrm{S}_N(X, x_0)$ and let $\sigma_N(X, x_0)$ be the set of equivalence classes.

The cardinality of the set $\sigma_N(X, x_0)$ is the desired invariant. It essentially determines the number of different ways of going to infinity in X. Since this cardinality depends on N, we have the following definition. For each integer $N > 0$ there is a function $\phi_N : \sigma_N(X, x_0) \to \sigma_{N+1}(X, x_0)$ that sends the equivalence class $[s]_N$ to the equivalence class $[s]_{N+1}$. X is said to be σ-stable if there is a $K > 0$ for which σ_N is a bijection for each integer $N \geq K$. If X is σ-stable let $\sigma(X, x_0)$ denote the cardinality of $\sigma_K(X, x_0)$.

It would be better to call X "σ-stable with respect to x_0" since apparently this definition depends on basepoint. In fact it does not; this issue is addressed in the next section.

The following is the main theorem of [1]. It is the theorem that we wish to extend to coarse equivalences.

Theorem 1 ([1], Theorem 3.2)**.** *Suppose $f : X \to Y$ is a bornologous equivalence between metric spaces. Let x_0 be a basepoint of X and set $y_0 = f(x_0)$. Suppose X and Y are σ-stable. Then $\sigma(X, x_0) = \sigma(Y, y_0)$.*

Change of basepoint in σ-stable spaces

As mentioned above, the definition of σ-stable depends on the choice of basepoint. We show that in fact a space being σ-stable is independent of basepoint.

Lemma 2. *Suppose $x_0, y_0 \in X$ and $n \geq d(x_0, y_0)$. Let $z_n : \sigma_n(X, x_0) \to \sigma_n(X, x_1)$ be the function that sends the equivalence class of a sequence $x_0, x_1, x_2, \ldots$ to the equivalence class of $y_0, x_0, x_1, x_2, \ldots$. Then z_n is a bijection.*

Proof. Let $w_n : \sigma_n(X, y_0) \to \sigma_n(X, x_0)$ be the function that sends the equivalence class of a sequence $y_0, y_1, y_2, \ldots$ to the equivalence class of $x_0, y_0, y_1, y_2, \ldots$. We show that z_n and w_n compose to form the identities and thus z_n must be a bijection. Suppose $[(x_i)] \in \sigma_n(X, x_0)$. Then $(w_n \circ z_n)([(x_i)])$ is the equivalence class of the sequence $x_0, y_0, x_0, x_1, \ldots$ which is a supersequence of (x_0). Similarly, $z_n \circ w_n$ is the identity on $\sigma_n(X, y_0)$. □

Proposition 3. *Suppose a metric space X is σ-stable with respect to a basepoint $x_0 \in X$. Let $y_0 \in X$. Then X is σ-stable with respect to y_0 and $\sigma(X, x_0) = \sigma(X, y_0)$.*

Proof. Let $N \in \mathbb{N}$ be such that $\phi_n : \sigma_n(X, x_0) \to \sigma_{n+1}(X, x_0)$ is a bijection for all $n \geq N$. Choose $M \in \mathbb{N}$ such that $M \geq \max\{N, \mathrm{d}(x_0, x_1)\}$. Suppose $n \geq M$. Then the following diagram commutes.

$$\begin{array}{ccc} \sigma_{n+1}(X, x_0) & \xrightarrow{z_{n+1}} & \sigma_{n+1}(X, y_0) \\ \uparrow{\scriptstyle \phi_n} & & \uparrow{\scriptstyle \psi_n} \\ \sigma_n(X, x_0) & \xrightarrow{z_n} & \sigma_n(X, y_0) \end{array}$$

Since ϕ_n, z_n, and z_{n+1} are bijections, so is ψ_n. □

The invariant

Theorem 4. *Suppose X and Y are coarsely equivalent and σ-stable. Then $\sigma(X) = \sigma(Y)$*

Proof. Suppose $f : X \to Y$ and $g : Y \to X$ are as in the definition of coarse equivalence. Let x_0 be a basepoint in X and $y_0 = f(x_0)$ be a basepoint in Y. Because $g \circ f$ is close to id_X, we can say that there is a D such that $\mathrm{d}(x, g \circ f(x)) \leq D$ for all $x \in X$. Let K be the integer provided by X being σ-stable and K' be the integer provided by Y being σ-stable. We can assume $K \geq D$. As f is bornologous, there is an M such that f sends K-sequences to M-sequences in Y. We can assume $M \geq K'$. Similarly, because g is bornologous, there is an L such that g sends M-sequences to L-sequences in X. We can assume $L \geq K$.

Let $z_L : \sigma_L(X, x_0) \to \sigma_L(X, g \circ f(x_0))$ be the function that sends the equivalence class of $x_0, x_1, \ldots$ to the equivalence class of $g \circ f(x_0), x_0, x_1, \ldots$. Note the latter is an L-sequence since $L \geq K \geq D$. By Lemma 2 we know z_L is a bijection.

Let f_K be the function that sends an element $[s]_K \in \sigma_K(X, x_0)$ to the element $[f(s)]_M \in \sigma_M(Y, f(x_0))$ and let g_M be the function that sends an element $[s]_M \in \sigma_M(Y, f(x_0))$ to the element $[g(s)]_L \in \sigma(X, g \circ f(x_0))$.

We show the following diagram commutes:

$$\begin{tikzcd}
\sigma_L(X, g \circ f(x_0)) & \\
\sigma_L(X, x_0) \arrow[u, "z_L"] & \\
\sigma_K(X, x_0) \arrow[u, "\phi_{KL}"] \arrow[r, "f_K"'] & \sigma_M(Y, f(x_0)) \arrow[uul, "g_M"']
\end{tikzcd}$$

Let (x_n) be a K-sequence in X. Then $g_M \circ f_K([(x_n)])$ is the equivalence class of the sequence $g \circ f(x_0), g \circ f(x_1), \ldots$ and $z_L \circ \phi_{KL}([(x_n)])$ is the equivalence class of the sequence $g \circ f(x_0), x_0, x_1, \ldots$. Consider the sequence

$$g \circ f(x_0), x_0, x_1, g \circ f(x_1), g \circ f(x_2), x_2, x_3, g \circ f(x_3), g \circ f(x_4), \ldots.$$

There are three distances to consider: the distance between successive elements of x_n, the distance between successive elements of $g \circ f(x_n)$, and the distance between any x_i, and its counterpart $g \circ f(x_i)$. Because $\mathrm{d}(x_i, g \circ f(x_i)) \leq D$, $\mathrm{d}(x_i, x_{i+1}) \leq K$, and $\mathrm{d}(g \circ f(x_i), g \circ f(x_{i+1})) \leq L$, the unioned sequence is an L-sequence. Further, because the two sequences $\{x_n\}$ and $\{g \circ f(x_n)\}$ are visited in order, we can say that x_n and $g \circ f(x_n)$ are both subsequences of this union. Thus, the diagram commutes.

Since $z_L \circ \phi_{KL}$ is a bijection, f_K must be one-to-one.

Symmetrically we can see that the following diagram commutes where S is chosen so that $\mathrm{d}(y, f \circ g(y)) \leq S$ for all $y \in Y$ and $\mathrm{d}(f(x), f(y)) \leq S$ whenever $\mathrm{d}(x, y) \leq L$.

$$
\begin{tikzcd}
 & \sigma_S(Y, f \circ g \circ f(x_0)) \\
 & \sigma_S(Y, f(x_0)) \arrow[u, "z_S"'] \\
\sigma_L(X, g \circ f(x_0)) \arrow[uur, "f_L"] & \sigma_M(Y, f(x_0)) \arrow[l, "g_M"] \arrow[u, "\psi_{MS}"']
\end{tikzcd}
$$

Thus g_M must be one-to-one which forces f_K to be onto. Then we have that f_K is a bijection.

$\square$

Some examples

We begin with the standard example of a coarse equivalence.

Example 5. *[2] Consider $\mathbb{R}$ and $\mathbb{Z}$ as metric spaces under the usual metric. Let $f : \mathbb{R} \to \mathbb{Z}$ be the floor function, $x \mapsto \lfloor x \rfloor$. Let $g : \mathbb{Z} \to \mathbb{R}$ be the inclusion, $n \mapsto n$. It is easy to see that f and g are coarse and that $g \circ f$ and $f \circ g$ are close to the identities ($g \circ f$ is the identity). Corollary 15 in [1] says that $\sigma(\mathbb{R}) = 2$. Since $\mathbb{Z}$ is coarsely equivalent to $\mathbb{R}$ we must have $\sigma(\mathbb{Z}) = 2$ also. Of course we can see these two sequences in $\mathbb{Z}$.*

Next we give another way to calculate $\sigma(V)$ where V is the vase from [1, Example 3]. We first give a basic lemma.

Lemma 6. *Suppose $f : X \to Y$ is any function and $g : Y \to X$ is bornologous. Suppose that $g \circ f$ is close to the identity on X. Then f is proper.*

Proof. Suppose $A \subset Y$ is bounded, say $\mathrm{d}(x, y) \leq N$ for all $x, y \in A$. Since g is bornologous there is an $M > 0$ so that if $\mathrm{d}(x, y) \leq N$, $\mathrm{d}(g(x), g(y)) \leq M$. Since $g \circ f$ is close to the identity there is an $R > 0$ so that $\mathrm{d}(g \circ f(x), x) \leq R$ for all $x \in X$.

Suppose $x, y \in f^{-1}(A)$. Then $f(x), f(y) \in A$ so $\mathrm{d}(f(x), f(y)) \leq N$ and $\mathrm{d}(g \circ f(x), g \circ f(y)) \leq M$. Thus $\mathrm{d}(x, y) \leq \mathrm{d}(x, g \circ f(x)) + \mathrm{d}(g \circ f(x), g \circ f(y)) + \mathrm{d}(g \circ f(y), y) \leq M + 2R$. $\square$

Example 7. *Let $V = \{(-1, y) : y \geq 1\} \cup \{(x, 1) : -1 \leq x \leq 1\} \cup \{(1, y) : y \geq 1\} \subset \mathbb{R}^2$. Following [1] we will use the taxicab metric which is bornologously (and therefore coarsely) equivalent to the standard metric. We show that V is coarsely equivalent to the ray $[1, \infty)$ and therefore $\sigma(V) = \sigma[1, \infty)$.*

First let us note that $\sigma[1, \infty)$ is fairly easy to calculate. Suppose $N \geq 1$. By [1, Lemma 2.4] an N-sequence $s = \{s_i\}$ in $[1, \infty)$ goes to infinity if and only if $\lim_{i \to \infty} s_i = \infty$. In particular $[(i)] \in \sigma_N[1, \infty)$. Also, given $[s] \in \sigma_N[1, \infty)$, since $\lim_{n \to \infty} s_i = \infty$ we see that s is equivalent to an increasing N-sequence t and if we put t and (i) together using the order on $\mathbb{R}$ we obtain the desired equivalence between s and (i). We have shown that $\sigma_N[1, \infty) = \{[(i)]\}$ so $\sigma[1, \infty) = 1$.

Now we define a coarse equivalence between V and $[1,\infty)$. Define $f : V \to [1,\infty)$ to send a point $(x,y) \in V$ to $y \in [0,\infty)$. Let $g : [1,\infty) \to V$ send $y \in [1,\infty)$ to $(1,y) \in V$. We have that $f \circ g(y) = y$ for all $y \in [1,\infty)$ so $f \circ g$ is the identity. Given $(x,y) \in V$, $d(g \circ f(x,y),(x,y)) = |x-1| + |y-y| \leq 2$ so $g \circ f$ is close to the identity.

By the lemma we need only to check that f and g are bornologous. Suppose $N > 0$, $(x_1,y_1),(x_2,y_2) \in V$, and $d((x_1,y_1),(x_2,y_2)) \leq N$. Then

$$\begin{aligned} d(f(x_1,y_1),f(x_2,y_2)) &= |y_1 - y_2| \\ &= d((x_1,y_1),(x_2,y_2)) - |x_1 - x_2| \leq N. \end{aligned}$$

Now suppose $y_1, y_2 \in [1,\infty)$ and $|y_1 - y_2| \leq N$. Then

$$d(g(y_1),g(y_2)) = d((1,y_1),(1,y_2)) = |1-1| + |y_1 - y_2| \leq N.$$

Proposition 8. *Let V be the vase from the previous example. Then V is not coarsely equivalent to $\mathbb{R}$.*

Proof. According to the previous example $\sigma(V) = 1$ and according to [1, Corollary 15] $\sigma(\mathbb{R}) = 2$. □

References

[1] B. Miller, L. Stibich, and J. Moore, *An invariant of metric spaces under bornologous equivalences*, Mathematics Exchange. **7** (2010) 12–19.

[2] J. Roe, Lectures on coarse geometry, University lecture series, **31** American Mathematical Society, Providence, RI, 2003.