Ball State Undergraduate Mathematics Exchange
https:// digitalresearch.bsu.edu/mathexchange
Vol. 15, No. 1 (Fall 2021)
Pages 2 – 8

A 21-Vertex 4-Chromatic Unit-Distance Graph of Girth 4

Daniel Kiteck , Kourtney Payne*

Daniel Kiteck received his Ph.D. in mathematics from the University of Kentucky in 2008. He has since enjoyed teaching at Indiana Wesleyan University, where he loves working with undergraduates in math research.

Kourtney Payne graduated Summa Cum Laude from Indiana Wesleyan University in 2017 with a Bachelor's degree in mathematics. She currently works as a senior physician compensation analyst at Sullivan Cotter in Southfield, Michigan.

Abstract The race to find the smallest 4-chromatic unit-distance graph of girth 4 stalled at 23 vertices in 1996. Using similar ideas to the 23-vertex graph, we constructed a 21-vertex graph. Unknown to us, the smallest possible of 17 vertices had already been created, but using a different approach. This paper carefully constructs our novel 21-vertex graph, while also comparing it to the 1996 23-vertex graph. We also give an overview of the construction of the 17-vertex graph.

1 Introduction

What is the smallest number of colors needed to color the points on the plane so that no two points at a unit-distance from each other have the same color? This smallest number is referred to as the *chromatic number of the plane*. Finding its value is a prominent open problem over a half-century old (search for *Hadwiger-Nelson Problem*). The only possibilities are five, six or seven colors. Four colors was eliminated as an option in 2018 when a unit-distance graph was found that is 5-chromatic, or, in other words, requires 5 colors to keep two adjacent vertices from having the same color [1]. Seven colors can be seen as an upper bound by coloring a regular-hexagon tiling of the plane in the following manner.

***Corresponding author**: daniel.kiteck@indwes.edu

Take a hexagon and its surrounding six hexagons and color them with seven different colors. Cover the plane by repeating this seven-color block. Depending on the given unit-distance, one can scale the colored tiling so that any two points a unit-distance apart will be different colors.

In thinking on the chromatic number of the plane, in 1975, Paul Erdős (of "Erdős number" fame) wondered if 4-chromatic unit-distance graphs without 3-cycles (or "triangle-free" or "of girth 4") exist:

> Let $\mathscr{S}$ be a subset of the plane which contains no equilateral triangle of size 1. Join two points of $\mathscr{S}$ if their distance is 1. Does this graph have chromatic number three? [2]

When Erdős republished the problem in 1979, he said the "...chromatic number is probably at most 3, but I do not see how to prove this." [3] Uncharacteristic for Erdős, he predicted incorrectly. In 1979, Nicholas Wormald showed that such a graph does indeed exist by publishing a 4-chromatic unit-distance 6448-vertex graph without 3-cycles (and, in fact, without 4-cycles) [4].

Alexander Soifer felt like 6448 vertices were a lot. So, in 1992, Soifer informally asked for the smallest example of a 4-chromatic unit-distance graph without 3-cycles [5] (p. 41, 110). From 1994 to 1996, three mathematicians accepted the challenge. Soifer says "A true World Series played out on the pages of *Geombinatorics...* ," a new journal Soifer had recently started [5] (p. 41, 42). First, the size was greatly reduced to 56 vertices by Paul O'Donnell in 1994 [6]. O'Donnell put two 5-pointed stars on a regular decagon, and then carefully connected seven 5-cycles in his construction. Next, Kiran Chilakamarri further reduced the size to 47 vertices in January 1995 [7], using a very different graph than O'Donnell's; Soifer called this the "Moth Graph" [5] (p. 118). Rob Hochberg improved the record by one (to 46), but he did not publish this since he heard about an even better result about to be published [5] (p. 125). The better result was, in July 1995, O'Donnell regaining first place with 40 vertices [8]. This graph has five-fold rotational symmetry. But then, O'Donnell and Hochberg combined forces to make the "Fish Graph" with an impressive 23 vertices in April 1996 [9] (Figure 1). This was the record for two decades. The details and pictures of the graphs of this "World Series" are recorded in chapter 15 of Soifer's book *The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of Its Creators* [5]. (This book also states many related open problems.)

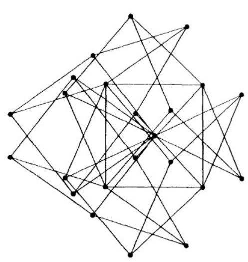

Figure 1: 23-Vertex "Fish Graph" (1996)

We, the authors of the article you are currently reading, thought we had found the first improvement since the Fish Graph. But only after we had finished our research, we realized that a 17-vertex graph (the smallest possible) had been found in 2016 by Geoffrey Exoo and Dan Ismailescu [10] (Figure 4). The following presents a novel approach to lower the 1996 record by two by extending the ideas of the Fish Graph. The construction of the Fish Graph will also be reviewed. An overview of the smallest graph possible, a 17-vertex graph, is given near the end of this article.

2 Construction of the 21-Vertex Graph

We found a 22-vertex graph (Figure 2) but realized we could coincide the two highlighted vertices to make a 21-vertex graph. The construction of the 21-vertex graph also shows how the 22-vertex graph is constructed.

Begin with a unit square WXYZ (Figure 3a). Add on V1, S and T so two rhombuses extend from the unit square (Figure 3b). Note that S has freedom to move while still keeping the graph unit-distance.

Construct pentagon Q1-Q2-Q3-Q4-Q5 where all edges are unit-distance, except possibly Q1-Q5, along with unit attachments Q1-V1, Q1-T, Q2-X, Q3-Z, Q4-X, and Q5-Z (Figure 3c). Moving S keeps everything unit-distance while taking Q1-Q5 from less than one unit to more than one unit. Fix S so that Q1-Q5 is unit-distance. This uses the argument of the Intermediate Value Theorem; this argument can be found in greater detail in [4] and [8].

The notation "X-Y→Z" is used to mean that Z is constructed to be unit-distance from both X and Y, such that X-Y-Z-X is counter-clockwise.

Select U2 unit-distance from Y. Then do the following construction, where the U's are vertices of the first pentagon of the Fish Graph (see below): U2-W→U3, Y-U3→U4, W-U2→U1, and U4-U1→U5 (Figure 3d). This construction gives U2 lots of freedom: the pentagon U1-U2-U3-U4-U5 is guaranteed to be unit-length with unit attachments for a range of choices for U2.

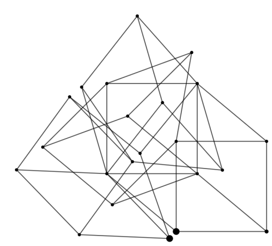

Figure 2: 22-Vertex

We next construct the second pentagon of the Fish Graph (see below), indicated with V's. Start with V1 that is already constructed. Then do V1-Y→V5, V5-U5→V4, W-V4→V3, and U5-V3→V2. Finally, connect V2 to V1 to complete the pentagon, but V2-V1 may not be unit-distance (Figure 3e). Thus, we need to make V2-V1 unit-distance. As U2 varies, the length of V2-V1 goes from below one unit to above one unit. Thus, fix U2 so that V2-V1 is one unit (again using the Intermediate Value Theorem). This finishes the construction of the 21-vertex graph. (Figure 3f).

Now a note on the construction of the Fish Graph. The Fish Graph has the starting square, but it does not have the two rhombuses or the Q-pentagon (the first pentagon constructed). Instead, immediately from the starting square, the U-pentagon and V-pentagon are constructed identically as given (except the vertex V1 will only be in the V-pentagon, and not overlap a vertex since the Fish Graph does not have the rhombuses). Then, to complete the Fish Graph, copies of the two pentagons (U and V) are flipped about the horizontal line through the center of the starting square, and the two pentagon copies are connected in the same way to the starting square as the two pentagon originals, just now "upside-down." This completes the construction of the Fish Graph and explains its horizontal line of symmetry (Figure 1) [9]. One vertex coincides with both pairs of pentagons, making 23 vertices instead of 24.

Here is a proof that the new 21-vertex graph is 4-chromatic. The proof uses elements

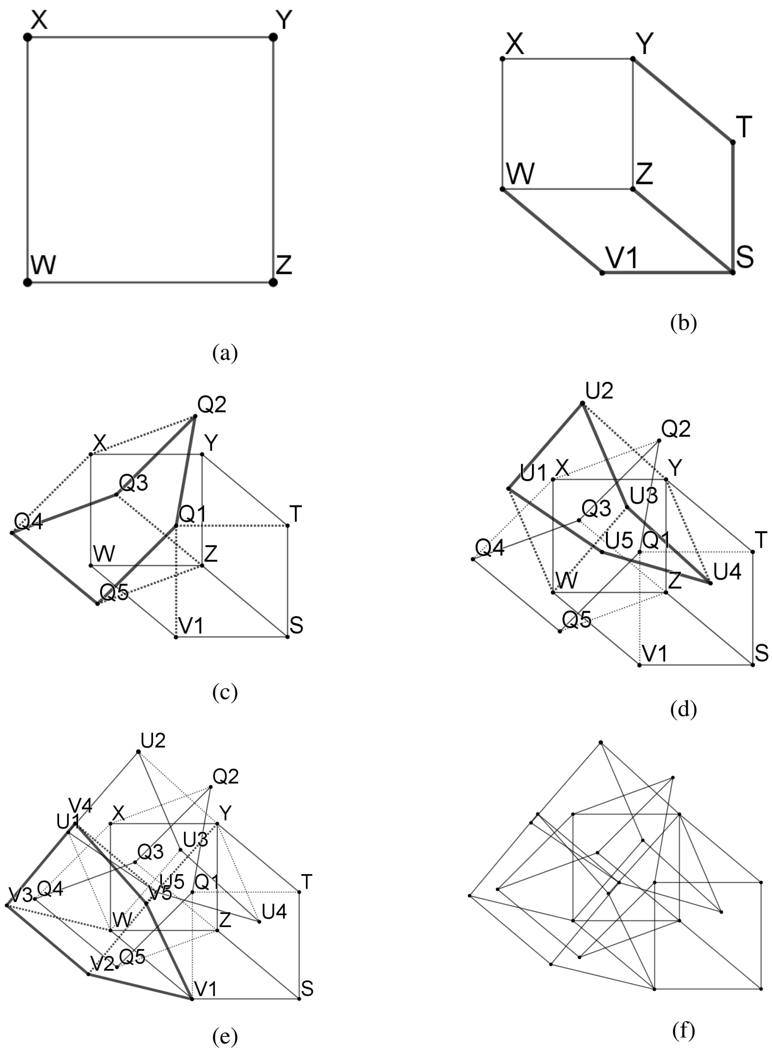

Figure 3: Constructing 21-Vertex Graph

from the proof for the Fish Graph (see below). In any attempted 3-coloring of the 21-vertex graph, one pair of diagonal vertices of the starting unit square must be the same color. Suppose W and Y are the same color, say green. It follows that U5 must also be green. Regardless of how the remaining vertices are colored, every vertex of the V-pentagon is now attached to a green vertex, so the V-pentagon can only be colored with two colors different than green, say blue and red. But pentagons are 3-chromatic, so a fourth color must be introduced.

Now suppose X and Z are the same color, say green. Let W be red and Y be blue. It follows that at least one of V1 and T must be green since otherwise S would be adjacent to three different colors with V1, Z, and T, so S would be forced to be a fourth color.

Then all five vertices of the Q-pentagon are attached to green vertices, so green cannot be used to color the Q-pentagon. But since pentagons are 3-chromatic, a fourth color must be introduced.

For the Fish Graph, the two pairs of pentagons (original U and V and the copies of U and V) work the same way on the vertices of the starting square as the U/V pair and Q pentagon do in the 21-vertex graph, making the Fish Graph 4-chromatic as well.

3 The Smallest Possible: a 17-Vertex Graph Found by Exoo and Ismailescu (2016)

As mentioned, we, the authors, were initially unaware that the 1996 record of 23 vertices in the Fish Graph had already been bettered in 2016. In fact, the new record of 17 vertices has been shown to be the smallest possible 4-chromatic unit-distance graph without 3-cycles (Figure 4). The full construction by Geoffrey Exoo and Dan Ismailescu is in [10]. The following gives an overview.

In a graph, a set of vertices is called *independent* if no two vertices in the set are adjacent. For their starting strategy, Exoo and Ismailescu say "The crucial idea of our approach is summarized in the two paragraphs below."

> Let G be a triangle-free 3-chromatic unit distance graph. For a given proper 3-coloring of the vertices, and a given independent set I, we say that I is *monochromatic* if all vertices of I receive the same color.
>
> Let $\mathscr{I}$ be a collection of independent sets of size 3 such that for every proper 3-coloring of G there exists a set $I \in \mathscr{I}$ which is monochromatic. It is then sufficient to attach 5-cycles **only** to the independent sets from $\mathscr{I}$, and the resulting graph will still be 4-chromatic. [10](p. 52)

This is a generalization of the technique used in the Fish Graph and this paper's 21 vertex graph: A set of pentagon(s) ("5-cycles") is attached in a manner where any proper coloring would have them attached to a monochromatic set, thus forcing another color, so pushing the graph from 3-chromatic to 4-chromatic.

Exoo and Ismailescu then use this method to construct a desired 21-vertex graph (distinctly different from the 21-vertex graph given in this paper– the 21-vertex graph by Exoo and Ismailescu has no connection to the Fish Graph). They start with an 11-vertex graph, and using a computer program, show that there is the desired collection of two independent sets (that at least one of the sets is monochromatic for any proper coloring). Then, laying the 11-vertex graph on a coordinate plane, they use computation technology to solve a system of non-linear equations that come from the restrictions of the graph to find the proper 5-cycles to attach, and where to attach them. Since two 5-cycles are attached, the graph is pushed up from 11 to 21 vertices. They then immediately better the 21 vertices. They show that one can find a starting graph with only **one** independent set that must be monochromatic regardless of coloring. The starting graph has 14 vertices, so when the 5-cycle is attached, the resulting final graph has only 19 vertices. This can be shown to be the smallest such graph using the strategy of independent sets and attaching 5-cycles. Exoo and Imailescu then wondered if they could be close enough to use computation technology to find any smaller, even the smallest.

They searched for graphs of order *n* that satisfy the following properties:

- 4-chromatic and *edge critical*, that is removal of any edge produces a graph which is 3-colorable.
- triangle-free and contain no forbidden subgraph of order up to 7 inclusive [10] (p. 61)

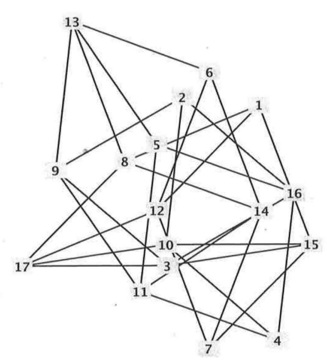

Figure 4: 17-Vertex: a smallest possible triangle-free 4-chromatic unit-distance graph, by Exoo and Ismailescu in 2016; picture from [10] p. 63

They did not elaborate on how they searched for these graphs. From these graphs, they then determined which could be unit-distance. They found none with less than 16 vertices. They found one with 16 vertices, but it did not work because there were two places where there was a unit-distance between a pair of vertices, and when the edges were filled in, it caused triangles. There were no others with 16 vertices. But they found one with 17 vertices. To show that a unit-distance embedding existed, they solved a non-linear polynomial system of six equations and six unknowns. "This system has 48 real solutions...which translates into 12 different embeddings discounting symmetries" [10] (p. 61). Since any smaller graphs would have shown up in their exhaustive list, 17 vertices must be the smallest possible.

The question still remains of what is the smallest 4-chromatic unit-distance graph with no 3-cycles *and* no 4-cycles. The smallest known 4-chromatic unit-distance graph with no 3- or 4-cycles is with 45 vertices [9]. But is this the smallest? This can be extended for larger cycles. It is hoped that considerations along these lines might help solve the chromatic number of the plane problem.

The authors thank Robert Hochberg for pointing us in the right direction going from 22 vertices to 21 vertices.

Bibliography

[1] Aubrey D.N.J. de Grey. "The Chromatic Number of the Plane Is at Least 5". Geombinatorics,XXVIII 1, 18-31, 2018

[2] Paul Erdős. "Unsolved Problems". Congressus Numerantium XV: Proceedings of the 5th British Combinatorics Conference 1975,681, 1976

[3] Paul Erdős "Combinatorial problems in geometry and number theory". Proceedings of Symposia in Pure Mathematics, XXXIV, American Mathematical Society: Relations between Combinatorics and Other Parts of Mathematics. Providence, RI, 149-162, 1979

[4] Nicholas C. Wormald. "A 4-Chromatic Graph with a Special Plane Drawing". Journal of the Australian Mathematical Society (Series A),28,1-8, 1979.

[5] Alexander Soifer. "The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of Its Creators". New York, NY: Springer New York,2009.

[6] Paul O'Donnell. "A Triangle-Free 4-Chromatic Graph in the Plane". Geombinatorics IV 1, 23-29, 1994.

[7] K. B. Chilakamarri. "A 4-Chromatic Unit Distance Graph with no triangles". Geombinatorics,IV 3, 64-76, 1995

[8] Paul O'Donnell. "A 40-Vertex 4-Chromatic Triangle-Free Unit Distance Graph". Geombinatorics V 1, 30-34, 1995.

[9] Robert Hochberg, Paul O'Donnell. "Some 4-Chromatic Unit-Distance Graphs Without Small Cycles". Geombinatorics V 4, 134-141, 1996.

[10] Geoffrey Exoo and Dan Ismailescu. "Small Order Triangle-Free 4-Chromatic Unit Distance Graphs". Geombinatorics XXVI 2, 49-64, 2016.