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Arithmetic Digit Manipulation and The Conway Base-13 Function

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Abstract

Despite positional notation being the primary way we represent numbers, it's not trivial to perform a variety of digit-manipulation with arithmetic alone. The Conway Base-13 Function is a prime example of a function who's definition is easily said in plain language, but difficult to formulate with arithmetic alone. To emphasize the difficulty, we construct a closed-form function equivalent to the Base-13 function over the integers, comprising only of arithmetic.

1 Introduction

Created by the great and late John H. Conway, the Conway Base 13 Function, $f:\mathbb{R}\to\mathbb{R}$, is a counterexample to the converse of the Intermediate Value Theorem. Despite f being discontinuous everywhere, it satisfies that for any interval (a,b), f takes all values between $f(a)$ and $f(b)$. In fact, f takes all values in $\mathbb{R}$ within every interval of non-zero length. Such a function can be defined in plain language in terms of digit-manipulation with relative ease, yet formulating f using arithmetic to perform such digit-manipulation is more difficult. Hence, the purpose of this article is to emphasize such difficulty by constructing a closed-form function equivalent to f over the integers, comprising only of arithmetic.

Imperatively, a summary of a definition will be given. Hence, let the set of digits in any base, $b\in\mathbb{Z}_{>1}$, be denoted

$$U_b=\{0,\dots,b-1\}.$$

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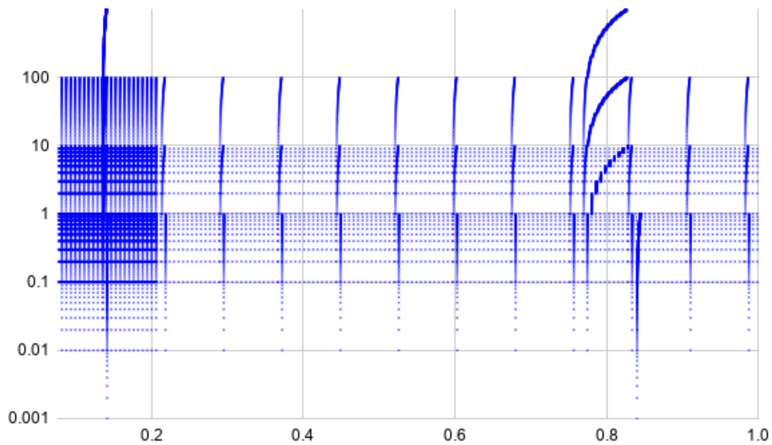

Figure 1: Log-plot of f over a subset of $\mathbb{Z}[\frac{1}{13}]$.

Thus, the set of decimal and tridecimal digits are $U_{10} = \{0,1,2,3,4,5,6,7,8,9\}$ and $U_{13} = \{0,1,2,3,4,5,6,7,8,9,A,B,C\}$, respectively. The digits A, B, and C correspond to their decimal equivalents 10, 11, and 12.

Suppose all $x \in \mathbb{R}_{\geq 0}$ have base-b expansions of the form

$$x = \dots d_1 d_0 . d_{-1} d_{-2} \dots_{(b)} \text{ s.t.} \sum_{k \in \mathbb{Z}} b^k d_k = x$$

where $d_k \in U_b$ are individual digits for all $k \in \mathbb{Z}$. Note that d_k corresponds to a digit to the left of the radix point only when $k \geq 0$. In reference to the position of a digit, the term *index* is used. A digit at the k^{th} index of an expansion refers to the digit k positions to the left of the units' column. Hence, a digit at index 0 is a digit in the units column. If no digit appears at the k^{th} index, the digit is assumed to be zero. Considering that some values of x and b have two expansions (such as in the cases $0.\overline{9}_{(10)} = 1_{(10)}$ or $1.2A\overline{C}_{(13)} = 1.2B_{(13)}$), we'll assume the terminating expansion is always preferred. For brevity, we'll introduce the notation $d_{j \to k(b)}$ as shorthand for $d_j d_{j-1} \dots d_{k+1} d_{k(b)}$. Furthermore, let $d_{j \to k(b)} \subseteq x$ represent, disregarding sign and radix point, that the sequence of digits $d_{j \to k(b)}$ occurs in the base-b expansion of x. For example

$$ABC_{(13)} \subseteq -A.BC_{(13)}.$$

If $x \in \mathbb{Z}_{\geq 0}$, then $k < 0 \implies d_k = 0$. Hence, for non-negative integer values, the base-b expansion of x can simply be written $d_{m \to 0(b)}$, where $m \in \mathbb{Z}_{\geq 0}$ is the largest index such that $d_m \neq 0$ (assuming $x \neq 0$, otherwise $m = 0$).

Adapted from a definition by Greg Oman [1], the Base-13 function f is defined in plain language as follows:

For any $x \in \mathbb{R}$, $k \in \mathbb{Z}$, let d_k represent the digit at index k in the tridecimal expansion of $|x|$. A few cases are considered:

- **Case 1:** Suppose there exists a digit $A \subseteq |x|$, such that all digits to the right of such do not contain A or B, and there exists exactly one $C \subseteq |x|$ to the right of such A. Let the digits between A and C be denoted $d_{j_A-1\to j_C+1}$, where j_A and j_C are the respective indices of such A and C. Let the digits after C be denoted $d_{j_C-1\to-\infty}$. Let $f(x) = +d_{j_A-1\to j_C+1}.d_{j_C-1\to-\infty(10)}$.
- **Case 2:** Suppose there exists a digit $B \subseteq |x|$, such that all digits to the right of such do not contain A or B, and there exists exactly one $C \subseteq |x|$ to the right of such B. Let the digits between B and C be denoted $d_{j_B-1\to j_C+1}$, where j_B and j_C are the respective indices of such B and C. Let the digits after C be denoted $d_{j_C-1\to-\infty}$. Let $f(x) = -d_{j_B-1\to j_C+1}.d_{j_C-1\to-\infty(10)}$.
- **Otherwise:** $f(x) = 0$ if x is not of either form.

Here, $d_{j_C-1\to-\infty}$ is shorthand for $\lim_{k\to-\infty} d_{j_C-1\to k(13)}$. It's important to recognize that the final result in cases 1 and 2 are decimal expansions, despite using digits from the tridecimal expansion of $|x|$. This is possible because in either case, the result only uses digits after the right-most A or B. Hence, the proceeding digits do not contain A or B. There's expectantly exactly one proceeding C, (the only other possible tridecimal digit which isn't also a decimal digit) however, which incidentally is excluded in the result. Hence, all digits in the result are indeed decimal. Essentially, f is a recompilation of some of the decimal digits in the tridecimal expansion of $|x|$, using a specific C (if it exists) as a decimal point, and A or B as the sign. Here are a few examples that cover all cases:

$$\begin{aligned}
f(-B1A.3C1415\ldots_{(13)}) &= \pi \\
f(137_{(13)}) &= 0 \\
f(0.B17C11_{(13)}) &= -17.11_{(10)} \\
f(0.\overline{A1C1}_{(13)}) &= 0 \\
f(0.A1\overline{C1}_{(13)}) &= 0 \\
f(0.A999C\overline{9}_{(13)}) &= 1000_{(10)}
\end{aligned}$$

It may be easy to see why f passes through all values of $\mathbb{R}$ within every non-zero-length interval. Regardless, proofs of its properties are not the purpose of this paper. Since the digit manipulation in f is not trivial, the ability to define f using only standard mathematical operations is not immediately clear.

Theorem 1. *There exists a closed-form $g : \mathbb{Z} \to \mathbb{R}$ such that $g \subseteq f$, where f is the Conway Base-13 Function.*

Understandably, such a prospect would benefit from quantifying its cases. The condition of the existence of a digit A or $B \subseteq |x|$, such that all digits to the right of such do not contain A or B can be quantified as $\exists j_A\big[\ d_{j_A} = A \wedge\ \nexists k < j_A(d_k \in \{A,B\})\big]$ or $\exists j_B\big[\ d_{j_B} = B \wedge\ \nexists k < j_B(d_k \in \{A,B\})\big]$ respectively. With the added condition that there exists exactly one $C \subseteq |x|$ to the right of such A or B, the cases become

$$\text{case 1} \iff \exists j_A\Big[\ d_{j_A} = A \wedge\ \nexists k < j_A(d_k \in \{A,B\}) \wedge \exists! j_C < j_A(d_{j_C} = C)\Big]$$

$$\text{case 2} \iff \exists j_B\Big[\ d_{j_B} = B \wedge\ \nexists k < j_B(d_k \in \{A,B\}) \wedge \exists! j_C < j_B(d_{j_C} = C)\Big]$$

This gives rise to an equivalent piecewise formulation:

$$f(x) = \begin{cases} +d_{j_A-1\to j_C+1}.d_{j_C-1\to-\infty(10)} & : \text{case } 1 \\ -d_{j_B-1\to j_C+1}.d_{j_C-1\to-\infty(10)} & : \text{case } 2 \\ 0 & : \text{otherwise} \end{cases}$$

2 Closed Form Expressions

As indicated and motivated by Nate Eldredge [4], a construction of f using only arithmetic functions is a possible procedure, albeit tedious and logic-heavy. It requires quite the array of functions designed to arbitrarily manipulate digits and test for logical conditions. This does not guarantee that the procedure will have a closed form over the entirety of $\mathbb{R}$, however it does give credence for a closed form over $\mathbb{Z}$.

2.1 Closed Form Operations

As there is no universal definition for closed-form expressions, we assume a conservative definition.

Definition 2. Let an operation be considered closed-form if it can be equivalently expressed in a finite number of operations, of which include addition, subtraction, multiplication, division, exponentiation, principal roots, and the principal branch of the logarithm.

This definition is restrictive so that operations that fulfill this conservative definition expectantly fulfill more liberal ones [2]. As evident in following sections, a significant number of arithmetic digit manipulation relies on the floor and ceiling operations. These can be defined through the use of their relationship to the modulo operation in floored division [3]:

$$\lfloor x \rfloor := x - (x \bmod 1),$$
$$\lceil x \rceil := x + ((-x) \bmod 1).$$

Here, mod is used as a binary operation as opposed to its use in congruence relations. It can be defined though the use of the periodic nature of the principal branch of the logarithm

$$x \bmod y := \frac{y}{2\pi i}\mathrm{Log}\left(e^{\frac{2\pi i x}{y}}\right)$$

assuming $0 \leq \frac{1}{i}\mathrm{Log}\left(e^{i\theta}\right) < 2\pi \ \forall\theta \in \mathbb{R}$. Hence, the floor, ceiling, and modulo operations will be considered closed-form. Similarly, the absolute value operation can be defined closed-form through the use of the principal square root, $|x| := \sqrt{x^2}$.

2.2 Logical-Conditional Functions

Given the natural piecewise definition of the Base-13 function, a multitude of functions that act for testing logical conditions are constructed. In particular, we construct functions that check for equality and inequality relations between two real numbers.

Definition 3. Let E, the "equivalence function", be defined as

$$E(a,b) := \lfloor (1+\varepsilon)^{-|a-b|} \rfloor \text{ such that } \varepsilon > 0,\ \forall a,b \in \mathbb{R}.$$

It is easily shown that

$$E(a,b) = \begin{cases} 1 & : a = b \\ 0 & : a \neq b \end{cases}$$

For brevity, the "negation" of the equivalence function will also be used.

Definition 4. Let N, the "non-equivalence function", be defined as

$$N(a,b) := 1 - E(a,b)\ \forall a,b \in \mathbb{R}.$$

Similarly,

$$N(a,b) = \begin{cases} 1 & : a \neq b \\ 0 & : a = b \end{cases}$$

Definition 5. Let G_E, the "greater-than or equal-to function", be defined as

$$G_E(a,b) := \left\lfloor \frac{1}{2} + \frac{1}{1+(1+\varepsilon)^{b-a}} \right\rfloor \text{ such that } \varepsilon > 0,\ \forall a,b \in \mathbb{R}.$$

Although not as trivial as the equivalence function, it can be shown that

$$G_E(a,b) = \begin{cases} 1 & : a \geq b \\ 0 & : a < b \end{cases}$$

Definition 6. Let M, the "minimum function", be defined as

$$M(a,b) := aG_E(b,a) + bG_E(a,b) - aE(a,b),\ \forall a,b \in \mathbb{R}.$$

By definition of G_E, it is clear that

$$M(a,b) = \begin{cases} a & : a \leq b \\ b & : a > b \end{cases}$$

These functions enable the ability to arithmetically test for logical conditions. With such, some digit manipulation that is naturally a more piecewise procedure, may instead be done entirely arithmetically.

3 Digit Manipulation

Singling-out digits from an expansion is the most critical ability of digit manipulation. As such, let us introduce the following closed-form functions:

Definition 7. Let $\overleftarrow{T}$, the "trailing-digit-truncation function", be defined as

$$\overleftarrow{T}_b^n(x) := \left\lfloor \frac{x}{b^n} \right\rfloor$$

$$\forall x \in \mathbb{Z}_{\geq 0}, \text{ for any base } b \in \mathbb{Z}_{>1}, \text{ and any digit-index } n \in \mathbb{Z}_{\geq 0}.$$

In essence, $\overleftarrow{T}$ removes the right-most n digits from a base-b expansion of x. More formally, it removes digits with indices less than a given index n.

Lemma 8. $x = d_{m\to 0(b)} \implies \overleftarrow{T}_b^n(x) = d_{m\to n(b)}$

Proof: Suppose $x = d_{m\to 0(b)}$. By definition of positional notation, $x = \sum_{k=0}^m b^k d_k$. Plugging this into $\overleftarrow{T}$ yields

$$\overleftarrow{T}_b^n(x) = \left\lfloor \frac{\sum_{k=0}^m b^k d_k}{b^n} \right\rfloor = \left\lfloor \sum_{k=0}^m b^{k-n} d_k \right\rfloor$$

which can be split into a whole and fractional part.

$$\begin{aligned}
&= \left\lfloor \sum_{k=n}^m b^{k-n} d_k + \sum_{k=0}^{n-1} b^{k-n} d_k \right\rfloor \\
&= \sum_{k=n}^m b^{k-n} d_k + \left\lfloor \sum_{k=0}^{n-1} b^{k-n} d_k \right\rfloor \\
&= \sum_{k=n}^m b^{k-n} d_k
\end{aligned}$$

We are left with a recompilation of the digits $d_{m\to n}$, such that d_n is now directly to the left of the radix point. In our notation, this is written $d_{m\to n(b)}$. □

For example, $\overleftarrow{T}_{10}^2(123456_{(10)}) = 1234_{(10)}$. In conjunction, the selection of an arbitrary digit at a given index is possible.

Definition 9. Let D, the "digit-selection function", be defined as

$$D_b^n(x) := \overleftarrow{T}_b^n(x) - b\overleftarrow{T}_b^{n+1}(x)$$

$$\forall x \in \mathbb{Z}_{\geq 0}, \text{ for any base } b \in \mathbb{Z}_{>1}, \text{ and any digit-index } n \in \mathbb{Z}_{\geq 0}.$$

This grants the ability to retrieve a digit at the n^{th} index of the base-b expansion of x within a closed-form manner. This ability is most critical in construction of the Base-13 Function.

Lemma 10. $x = d_{m\to 0(b)} \implies D_b^n(x) = d_n$

Proof: Suppose $x = d_{m\to 0(b)}$. Using Lemma 8, D becomes

$$\begin{aligned} D_b^n(x) &= \sum_{k=n}^{m} b^{k-n} d_k - b \sum_{k=n+1}^{m} b^{k-n-1} d_k \\ &= \sum_{k=n}^{m} b^{k-n} d_k - \sum_{k=n+1}^{m} b^{k-n} d_k \\ &= d_n + \sum_{k=n+1}^{m} b^{k-n} d_k - \sum_{k=n+1}^{m} b^{k-n} d_k \\ &= d_n \end{aligned}$$

□

For example, $D_{10}^2(123456_{(10)}) = 4_{(10)}$. Not surprisingly, the number of digits in an expansion can also be deduced arithmetically.

Definition 11. Let L, the "length function", be defined as

$$L_b(x) := \lceil \log_b(x+1) \rceil + E(x,0)$$

$$\forall x \in \mathbb{Z}_{\geq 0}, \text{ for any base } b \in \mathbb{Z}_{>1}.$$

This is variant of the usual method to count the number of digits: $\lfloor \log_b(x) \rfloor + 1$. However the latter is undefined for the case $x = 0$, whereas $L_b(0) = 1$. Otherwise both methods are equivalent over the positive integers.

Lemma 12. $x = d_{m\to 0(b)} \wedge x > 0 \implies L_b(x) = m+1$

Proof: Suppose $x = d_{m\to 0(b)} \wedge x > 0$,

$$\begin{aligned} &\implies L_b(x) = \left\lceil \log_b \left(1 + \sum_{k=0}^{m} b^k d_k\right) \right\rceil \\ \implies \left\lceil \log_b(b^m) \right\rceil < L_b(x) \leq \left\lceil \log_b(b^{m+1}) \right\rceil \\ &\implies m < L_b(x) \leq m+1 \\ &\implies L_b(x) = m+1 \end{aligned}$$

□

For example, $L_{10}(10_{(10)}) = L_{10}(99_{(10)}) = 2$. If $x \in \mathbb{Z}_{\geq 0}$ and $d \in U_b$, then functions D, E, L can be used to count the occurrences of d in the base-b expansion of x.

Definition 13. Let O, the "digit-occurrence-counting function", be defined as

$$O_b^p(x) := \sum_{k=0}^{L_b(x)-1} E(D_b^k(x), p)$$

$$\forall x \in \mathbb{Z}_{\geq 0}, \text{ for any base } b \in \mathbb{Z}_{>1}, \text{ and any digit } p \in U_b.$$

It should be evident that as O loops through all possible indices, k, for digits in the base-b expansion of x, the summation increments by 1 iff the digit at index k is equivalent to the given digit p, which we are looking to count the occurrences of. In other words, O counts the number of occurrences of a digit p in the base-b expansion of x.

Less trivial is a method to deduce the a specific index of an occurrence of a given digit.

Definition 14. Let I, the "digit-occurrence-index function", be defined as

$$I_b^p(x) := \sum_{k=1}^{L_b(x)} E\Big(O_b^p\big(\overleftarrow{T}_b^k(x)\big), O_b^p(x)\Big)$$

$$\forall x \in \mathbb{Z}_{\geq 0},\ \text{for any base } b \in \mathbb{Z}_{>1},\ \text{and any digit } p \in U_b.$$

The purpose of I is to return the index of the right-most digit p in the base-b expansion of x. If there isn't such an index, then I returns $L_b(x)$, which is by definition a number higher than the maximum index of a nonzero digit.

Lemma 15.

$$x = d_{m\to 0(b)} \implies I_b^p(x) = \begin{cases} j & : \exists j\big[d_j = p \wedge \forall k < j(d_k \neq p)\big] \\ L_b(x) & : \text{otherwise} \end{cases}$$

Proof: Suppose $x = d_{m\to 0(b)}$. We'll look at the case where there does exist a right-most digit p in the base-b expansion of x.

Case 1: $\exists j\big[d_j = p \wedge \forall k < j(d_k \neq p)\big]$

Thus, with such a digit having index j, truncating off digits of x with indices less than k for $k \leq j$, yields a number with no occurrences of p removed. Likewise, truncating for $k > j$ yields a number with at least one less occurrence of p.

$$k \leq j \iff O_b^p\big(\overleftarrow{T}_b^k(x)\big) = O_b^p(x)$$
$$\implies E\Big(O_b^p\big(\overleftarrow{T}_b^k(x)\big), O_b^p(x)\Big) = \begin{cases} 1 & : k \leq j \\ 0 & : k > j \end{cases}$$

Thus, the summation can be split into

$$I_b^p(x) = \sum_{k=1}^{j} 1 + \sum_{k=j+1}^{L_b(x)} 0 = j$$

Resulting in the index, j.

Case 2: $\nexists j\big[d_j = p \wedge \forall k < j(d_k \neq p)\big]$

In the other case, since x is an integer of finite digits, there not being a right-most digit p implies that there are no occurrences.

$$\begin{aligned} O_b^p\big(\overleftarrow{T}_b^k(x)\big) &= O_b^p(x) = 0\ \forall k \\ \implies E\Big(O_b^p\big(\overleftarrow{T}_b^k(x)\big), O_b^p(x)\Big) &= 1 \\ \implies I_b^p(x) &= \sum_{k=1}^{L_b(x)} 1 \\ &= L_b(x). \end{aligned}$$

As such, the sum is trivially the bound, $L_b(x)$.

$$\text{Therefore } x = d_{m\to 0(b)} \implies I_b^p(x) = \begin{cases} j & : \exists j\big[d_j = p \wedge \forall k < j(d_k \neq p)\big] \\ L_b(x) & : \text{otherwise} \end{cases}$$

□

In parody to $\overleftarrow{T}$, we'll define a function that virtually removes the left-most n digits from a base-b expansion of x.

Definition 16. Let $\overrightarrow{T}$, the "leading-digit-truncation function", be defined as

$$\overrightarrow{T}_b^n(x) := \sum_{k=0}^{L_b(x)-n-1} b^k D_b^k(x)$$

$$\forall x \in \mathbb{Z}_{\geq 0},\ \text{for any base } b \in \mathbb{Z}_{>1},\ \text{for any digit-index } n \in \mathbb{Z}_{\geq 0}$$

Clearly, $\overrightarrow{T}$ reassembles the digits in the base-b expansion of x into their original position, save for the last n digits.

Definition 17. Let K, the "cut-to-index function", be defined as

$$K_b^p(x) := \sum_{k=0}^{I_b^p(x)} b^k D_b^k(x)$$

$$\forall x \in \mathbb{Z}_{\geq 0},\ \text{for any base } b \in \mathbb{Z}_{>1},\ \text{for any digit } p \in U_b.$$

Similar to $\overrightarrow{T}$, K reassembles the digits in the base-b expansion of x into their original position, save for the last digits with indices greater than $I_b^p(x)$. For the case where $p \nsubseteq x$, we find that $I_b^p(x) = L_b(x)$, which implies that $K_b^d(x) = x$.

4 Assembling The Conway Base-13 Function

Perhaps the most daunting of tasks to replicate in the Conway Base-13 Function is recompiling digits in an expansion from one base to another, and replacing a digit with a radix-point.

Definition 18. Let X, the "base-to-base re-radix function", be defined as

$$X^p_{b_1,b_2}(x) := \sum_{k=0}^{L_{b_1}(x)-1} N(D^k_{b_1}(x),p) D^k_{b_1}(x) b_2^{k-I^p_{b_1}(x)-G_E(I^p_{b_1}(x),k)}$$

$\forall x \in \mathbb{Z}_{\geq 0},$ for any bases $b_1, b_2 \in \mathbb{Z}_{>1},$ and any digit $p \in U_{b_1}.$

X removes a specific digit p, with index j in the base-b_1 expansion of x. This position will be virtually used as a new radix-point. Digits to the left of p (with indices $k > j$) are placed directly to left of this new radix, and digits to the right of p (with indices $k < j$) are placed directly to the right. The final result is treated as a base-b_2 expansion. For the instances where there are multiple occurrences of p, such a case is evidently disregarded in further construction of the Base-13 function.

Lemma 19.

$$x = d_{m\to 0(b_1)} \wedge \exists! j(d_j = p) \implies X^p_{b_1,b_2}(x) = d_{m\to j+1}.d_{j-1\to 0(b_2)} \ \forall b_2 \in \mathbb{Z}_{\geq b_1}$$

Proof: Suppose $x = d_{m\to 0(b_1)}$ and $\exists! j(d_j = p)$. Thus the index, j, is given by $I^p_{b_1}(x) = j$. A digit at index k is given by $D^k_{b_1}(x) = d_k$. Hence $\forall b_2 \in \mathbb{Z}_{\geq b_1}$, substituting for our positional notation,

$$N(D^k_{b_1}(x),p) D^k_{b_1}(x) b_2^{k-I^p_{b_1}(x)-G_E(I^p_{b_1}(x),k)} = \begin{cases} d_k b_2^{k-j} & \text{if } k < j \\ 0 & \text{if } k = j \\ d_k b_2^{k-j-1} & \text{if } k > j \end{cases}$$

which can be used to split the sum into

$$X^p_{b_1,b_2}(x) = \sum_{k=0}^{j-1} d_k b_2^{k-j} + \sum_{k=j+1}^{m} d_k b_2^{k-j-1}$$

We are left with two recompilations of digits from base-b_1 to base-b_2, with the digits to the left of p directly to left of the radix, and digits to the right of p to the right. In our positional notation, this is equivalent to $d_{m\to j+1}.d_{j-1\to 0(b_2)}$.

$\square$

For example, $X^C_{13,10}(1C3_{(13)}) = 1.3_{(10)}$. Lastly, we'll introduce a method to detect whether one of two given digits are contained within a base-b expansion. This will act as the step in determining if the final expansion of Conway's Base-13 function will be positive or negative.

Definition 20. Let S, the "resulting-sign function", be defined as

$$S^{p_1,p_2}_b(x) := E\big(O^{p_1}_b(x),1\big) - E\big(O^{p_2}_b(x),1\big)$$

$\forall x \in \mathbb{Z}_{\geq 0},$ for any base $b \in \mathbb{Z}_{>1},$ for any digits $p_1, p_2 \in \{0,\ldots,b-1\}.$

Unlike the previous function, S is much simpler in description. If there exists exactly one $p_1 \subseteq x$, and not exactly one $p_2 \subseteq x$ (assuming a base-b expansion), then $S_b^{p_1,p_2}(x) = 1$. Similarly, $S_b^{p_1,p_2}(x) = -1$ if there exists exactly one $p_2 \subseteq x$, and not exactly one $p_1 \subseteq x$. Otherwise the result is zero.

Lemma 21.

$$x = d_{m\to 0(b)} \implies S_b^{p_1,p_2}(x) = \begin{cases} +1 & : \exists! j_1 (d_{j_1} = p_1) \wedge \nexists! j_2 (d_{j_2} = p_2) \\ -1 & : \nexists! j_1 (d_{j_1} = p_1) \wedge \exists! j_2 (d_{j_2} = p_2) \\ 0 & : \text{otherwise} \end{cases}$$

Proof: Suppose $x = d_{m\to 0(b)}$. With the definitions of E and O, the values of S, defined by $E\left(O_b^{p_1}(x), 1\right) - E\left(O_b^{p_2}(x), 1\right)$, in the following case-table are straightforward.

cases	$\exists! j_1 (d_{j_1} = p_1)$	$\nexists! j_1 (d_{j_1} = p_1)$
$\exists! j_2 (d_{j_2} = p_1)$	$S_b^{p_1,p_2}(x) = 1 - 1 = 0$	$S_b^{p_1,p_2}(x) = 0 - 1 = -1$
$\nexists! j_2 (d_{j_2} = p_2)$	$S_b^{p_1,p_2}(x) = 1 - 0 = 1$	$S_b^{p_1,p_2}(x) = 0 - 0 = 0$

□

With an arsenal of closed-form logical-conditional and digit manipulating functions, the Conway Base-13 Function can be constructed.

Theorem 2 There exists a closed-form $g : \mathbb{Z} \to \mathbb{R}$ such that $g \subseteq f$, where f is the Conway Base-13 Function.

Proof. Let $f_1(x) = M\left(K_{13}^A|x|, K_{13}^B|x|\right)$. After applying f_1 to an integer x, digits directly to the left of the rightmost A or B in the tridecimal expansion of x are truncated. As the sign of the input is disregarded in the Base-13 Function, the absolute value of x is taken for each instance of x in f_1. For any $k \in \mathbb{Z}$, let d_k represent the digit at index k in the tridecimal expansion of $|x|$. Let the rightmost-index of A be written as $I_{13}^A|x| = j_A$ and the rightmost-index of B be written as $I_{13}^B|x| = j_B$. Note that by our definitions of I and K,

$$A \not\subseteq x \implies j_A = L_{13}|x| \implies K_{13}^A|x| = |x|$$

$$B \not\subseteq x \implies j_B = L_{13}|x| \implies K_{13}^B|x| = |x|$$

Since L is monotonically increasing, the inequality relation between j_A and j_B implies

$$j_A \le j_B \iff L_{13}\left(K_{13}^A|x|\right) \le L_{13}\left(K_{13}^B|x|\right) \iff K_{13}^A|x| \le K_{13}^B|x|$$

$$j_B \le j_A \iff L_{13}\left(K_{13}^B|x|\right) \le L_{13}\left(K_{13}^A|x|\right) \iff K_{13}^B|x| \le K_{13}^A|x|$$

Therefore, they determine the value of M by

$$j_A \le j_B \iff M\left(K_{13}^A|x|, K_{13}^B|x|\right) = K_{13}^A|x|$$

$$j_B \le j_A \iff M\left(K_{13}^A|x|, K_{13}^B|x|\right) = K_{13}^B|x|$$

Consequently f_1 becomes

$$f_1(x) = \begin{cases} K_{13}^{A}|x| & : j_A < j_B \\ K_{13}^{B}|x| & : j_B < j_A \\ |x| & : \text{otherwise} \end{cases}$$

And by definition of K, the function f_1 can be represented

$$f_1(x) = \begin{cases} Ad_{j_A-1\to 0(13)} & : A \subseteq x \wedge j_A < j_B \\ Bd_{j_B-1\to 0(13)} & : B \subseteq x \wedge j_B < j_A \\ |x| & : \text{otherwise} \end{cases}$$

Next, let $f_2(x) = f_1(x)E\Big(O_{13}^{C}\big(f_1(x)\big),1\Big)$. In f_2, we are checking if after such an A or B, there exists exactly one C leftover in the tridecimal expansion of $f_1(x)$. If there does not exist exactly one such C,

$$O_{13}^{C}\big(f_1(x)\big) \neq 1 \implies E\Big(O_{13}^{C}\big(f_1(x)\big),1\Big) = 0 \implies f_2(x) = 0$$

Otherwise, let the index of such be denoted $I_{13}^{C}\big(f_1(x)\big) = j_C$. Therefore

$$f_2(x) = \begin{cases} Ad_{j_A-1\to j_C+1}Cd_{j_C-1\to 0(13)} & : A \subseteq f_1(x) \wedge O_{13}^{C}(f_1(x)) = 1 \\ Bd_{j_B-1\to j_C+1}Cd_{j_C-1\to 0(13)} & : B \subseteq f_1(x) \wedge O_{13}^{C}(f_1(x)) = 1 \\ f_1(x) & : A,B \not\subseteq f_1(x) \wedge O_{13}^{C}(f_1(x)) = 1 \\ 0 & : O_{13}^{C}(f_1(x)) \neq 1 \end{cases}$$

Lastly, let $f_3(x) = S_{13}^{A,B}\Big(f_2(x)\Big)X_{13,10}^{C}\Big(\overrightarrow{T}^{1}_{13}\big(f_2(x)\big)\Big)$. This final function determines the sign of the final result, truncates off the leftover A or B, recompiles the tridecimal expansion into decimal, and essentially replaces C with a decimal point (assuming $C \subseteq f_2(x)$). By the definition of S and f_2, we find that

$$S_{13}^{A,B}\Big(f_2(x)\Big) = \begin{cases} +1 & : A \subseteq f_2(x) \\ -1 & : B \subseteq f_2(x) \\ 0 & : \text{otherwise} \end{cases}$$

and by our definition of $\overrightarrow{T}^{*}$,

$$\overrightarrow{T}^{1}_{13}\big(f_2(x)\big) = \begin{cases} d_{j_A-1\to j_C+1}Cd_{j_C-1\to 0(13)} & : A \subseteq f_1(x) \wedge O_{13}^{C}(f_1(x)) = 1 \\ d_{j_B-1\to j_C+1}Cd_{j_C-1\to 0(13)} & : B \subseteq f_1(x) \wedge O_{13}^{C}(f_1(x)) = 1 \\ d_{L_{13}|x|-2\to 0(13)} & : A,B \not\subseteq f_1(x) \wedge O_{13}^{C}(f_1(x)) = 1 \\ 0 & : O_{13}^{C}(f_1(x)) \neq 1 \end{cases}$$

Hence, by definition of X,

$$X^C_{13,10}\left(\overrightarrow{T}^1_{13}\left(f_2(x)\right)\right) =$$
$$\begin{cases} d_{j_A-1\to j_C+1}.d_{j_C-1\to 0(10)} & : A \subseteq f_1(x) \land O^C_{13}(f_1(x)) = 1 \\ d_{j_B-1\to j_C+1}.d_{j_C-1\to 0(10)} & : B \subseteq f_1(x) \land O^C_{13}(f_1(x)) = 1 \\ d_{L_{13}|x|-2\to j_C+1}.d_{j_C-1\to 0(10)} & : A,B \nsubseteq f_1(x) \land O^C_{13}(f_1(x)) = 1 \\ 0 & : O^C_{13}(f_1(x)) \neq 1 \end{cases}$$

Therefore, the final result is of form

$$f_3(x) = \begin{cases} +d_{j_A-1\to j_C+1}.d_{j_C-1\to 0\cdot(10)} & : A \subseteq f_2(x) \land C \subseteq f_2(x) \\ -d_{j_B-1\to j_C+1}.d_{j_C-1\to 0\cdot(10)} & : B \subseteq f_2(x) \land C \subseteq f_2(x) \\ 0 & : \text{otherwise} \end{cases}$$

It should be seen that when x is an integer, the result for each case in f_3 is equivalent to f, the Base-13 function. Furthermore, as the cases from our original quantification from the plain-language definition hold equivalently, we find that $f_3 \subseteq f$, directly satisfying that f_3 is a closed-form representation of the Conway Base-13 Function over the integers.

□

5 Concluding Remarks

Due to the fractal-like symmetry of f, such that $f(13^n x) = f(x) \forall n \in \mathbb{Z}$, it's possible to extend $f_3 : \mathbb{Z} \to \mathbb{R}$ to $f_3 : \mathbb{Z}[\frac{1}{13}] \to \mathbb{R}$ by imposing that if $x = \frac{y}{13^n} \forall y \in \mathbb{Z},\ \forall n \in \mathbb{Z}_{\geq 0}$, then $f_3(x) = f_3(y)$. This was done in the creation of Figure 1. It may be possible to extend f_3 to even larger sets of numbers whose distribution of digits are known, but a closed-form for f over the entirety of $\mathbb{R}$ is impossible, as the digit-distribution for every real number is not computable [4].

It is no doubt that the computational efficiency of these algorithms is far from optimal. A computer can perform a variety of digit manipulation tasks directly and quite efficiently, without the use of arithmetic closed-form functions such as these. The purpose, rather, was to fulfill the recreational endeavor of finding the first equation for Conway's Base-13 Function, based solely on finite arithmetic. This work was inspired by the recent passing of John H. Conway (1937→2020).

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Note that the value in the third case is irrelevant, as such a case results in 0 in f_3.

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