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Dynamic Optimization in Building Personal Emergency Fund

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Abstract

The 2018 National Financial Capability Study found that 46 percent of Americans do not have the recommended three months' worth of expenses in the case of an emergency. It is of immense importance to provide the best financial strategies towards building a solid financial foundation. In this paper, we examine how to build an emergency fund while maximizing the utility of consumption, allowing for a balance of consumer gratification and necessary future planning. This problem was approached utilizing the method of dynamic optimization. The necessary conditions for optimality were obtained and computations were performed to determine the optimal solution. The optimal savings trajectory was adjusted monthly by incorporating sensitivity factors with respect to each parameter involved in the model to get the actual monthly savings. Finally, we performed numerical simulations to create a financial plan that achieves a prescribed amount of emergency fund goal in a given planning year utilizing simulation data from an entry-level college graduate's salary, current high-yield return rates, and treasury yield-to-maturity rates.

1 Introduction

Everyday, Americans strive to achieve their "American Dream" which could be finding a better job, getting a better education, buying a property, etc. However, with the acquisition of a better life style, Americans' face greater financial responsibilities. Without an appropriate financial understanding, by the end of each paycheck, one's "American Dream" can easily turn into a nightmare. According to the 2018 National Financial Capability Study, the subject of personal finances is a source of anxiety. The study says, "more than half (53%) agree that thinking about their finances makes them anxious, and 44% feel that discussing their finances is stressful, with respondents ages 18-34 reporting the highest levels of stress (63%) and anxiety (55%)." With that being said, Bankrate, a personal finance website, conducted a survey that shows only 40% of Americans are comfortable covering $1,000 of unexpected expenses. Therefore, emergencies that might cost more, such as losing a job or getting injured, can be out of the question. After further research on the subject of personal finances, there is one common topic that is covered in almost every website and financial literacy book- the subject of building an emergency fund. An emergency fund is a saved currency that is easy to liquidate in case of emergencies. Having such a fund promises financial security. Despite the importance of having "rainy day" funds, the 2018 National Financial Capability Study found that 46% of Americans have not yet set aside funds enough to cover three months worth of expenses in case of an emergency. We feel it is of immense importance to provide a reference for those looking to get on the right track toward building a solid financial foundation.

Both savings and emergency funds bring positive changes into well-being of households. The first and most obvious benefit is interest that can be earned on money in savings accounts. Today most of the banks have low interest rates, however by investing into a high yield savings account (as we are going to present in our paper), one can maximize the interest that can be earned. Another great benefit is that there is basically no risk involved. Unlike, the stock market, which is considered a high-risk investment type, putting money in a savings account will not cause any losses. In addition, today most of the banks are insured, which ensures that one's money is safe in banks. Today's automatic deposits make it convenient to save without having to physically be present in banks. There are a lot more benefits that come with having extra money saved. Therefore, we hope that this paper will bring essential input on presenting how individuals can achieve their saving goals.

In this paper we attempt to present how Dynamic Programming and Optimal Control Theory can be applied in optimization models that deal with efficiency of savings and consumption. For simulation purposes we are assuming the year 2003 as a current time. The data that we have collected is assumed as a predicted data. The problem that we consider is a system that continuously evolves over time and we are looking for an optimal solution or trajectory for the state variable using dynamic optimization. While applications of dynamic programming and optimization are still new in economics, we believe that further work can lead to an improvement of the welfare of households. With this practical experiment we will implement mathematical techniques used in optimization. Our goal is to maximize the utility of consumption (which has an effect on overall satisfaction of a given person) while building up the savings fund. Solution to such a problem will involve structuring it into multiple stages that can be performed by using dynamic-programming approach. This approach also constitutes the states of the process. The state variable trajectory help us to evaluate future actions taken based on present decisions. Despite the rich theoretical concept behind our project, it will have a realistic connection because of the factual data, including treasury yield-to-maturity rates and treasury bond rates. We believe that this paper will bring further contributions in applications of mathematics and dynamic programming in fields of economics and finance.

2 Mathematical Model

Suppose a teacher wants to set up an emergency fund that will allow them to save a certain amount of money over a specific time period while also maximizing their utility. We seek an optimal balance between saving money while also having the most satisfaction from the money you are able to consume. Let $s(t)$ be the state of the savings account at time t and changes at the rate $\frac{ds}{dt}(t) = s'(t)$ with $s(0) = s_0$. Let $c(t)$ be the amount of money available for consumption at time t after saving. The teacher's total consumption at any moment t is

$$c(t) = E(t) + \rho(t)s(t) - s'(t),$$

where $E(t)$ is after tax earnings at time t and $\rho(t)s(t)$ is a return from the savings account. The natural log is frequently used in economics to capture the relationship between consumption and utility, $u(t) = \ln(c(t))$. It shows that the utility of each additional dollar of consumption declines as the level of consumption increases.

The teacher's objective is to determine the trajectory of the state, savings accumulation

$s(t)$, to maximize the functional

$$J = \int_0^T \ln(c(t))e^{-r(t)t}\,dt = \int_0^T \left[\ln(E(t) + \rho(t)s(t) - s'(t))\right] e^{-r(t)t}\,dt$$

subject to $s(0) = s_0 = 0$, $s(15) = 60,000$ and $s'(t) \geq 0$, where $\rho(t) =$ the return rate for the savings account at time t, $r(t)$ is the treasury yield-to-maturity rate at time t, $e^{-r(t)t}$ is the discount factor, and $\ln(c(t))e^{-r(t)t}$ is the present value of utility.

To formulate our problem in a control theory set up we summarize the state and control variables and parameters involved in our model as:

$$\begin{aligned}
c(t) &= \text{consumption at time t} \\
s(t) &= \text{the state of the savings account at time t} \\
r(t) &= \text{ treasury yield-to-maturity rate at time t} \\
E(t) &= \text{after tax earnings at time t} \\
\rho(t) &= \text{return rate for the savings account at time t} \\
s'(t) &= \text{what is being deposited into the savings at time t}
\end{aligned}$$

We assume that in each year the monthly salary to be a constant (which is also true in most professions), $E(t) = E$. We also assume that the return rate on savings and treasury yield-to-maturity rate (both depend on the market) are constant for short intervals of time by taking the average predicted values. Then we will perform sensitivity analysis of the optimal solution with respect to these parameters and adjust our solutions accordingly.

3 Mathematical Tools

Calculus of Variations (modern Optimal Control Theory) is used in mathematics to find minimums and maximums of functionals that involve functions that change over time. The origins are traced back to $1696 - 1697$ when John Bernoulli and his brother James were solving the brachistochrone problem. Later the search for the necessary conditions for an extremal to be a minimizer led to the development of the Euler-Lagrange equation. It was widely used in mathematics to solve problems of optimization, which led to fruitful outcomes in many fields such as aerospace engineering and machine learning. In early 1930 mathematicians and economic theorists such as Ramsey and Hotelling started developing optimization theories related to the field of economics (Kamien [1]). To find the optimal solution we will derive the Euler-Lagrange equation. Suppose that we have the functional

$$J(s(t)) = \int_0^T F(t, s(t), s'(t))\,dt.$$

We wish to find a function $s(t)$ that satisfies the boundary conditions $s(0) = 0$ and $s(T) = s_T$ and maximizes the functional J. Suppose that $s^*(t)$ is such a function. Then any small perturbations of $s^*(t)$ that preserves the boundary conditions will decrease the value of J since $s^*(t)$ is a maximizer.

Let $h(t)$ be a function that is continuous and differentiable on $[0,T]$ and $(0,T)$, respectively, such that $h(0) = h(T) = 0$ and let $\varepsilon \in \mathbb{R}$. Then define

$$J(\varepsilon) = \int_0^T F\left(t, s^*(t) + \varepsilon h(t), s^{*'}(t) + \varepsilon h'(t)\right) dt$$

to be the resulting functional under the slight perturbations. We wish to find the total derivative of $J(\varepsilon)$ with respect to ε. Therefore we have

$$\begin{aligned}\frac{d}{d\varepsilon}(J(\varepsilon)) &= \frac{d}{d\varepsilon}\int_0^T F\left(t, s^*(t) + \varepsilon h(t), s^{*'}(t) + \varepsilon h'(t)\right) dt \\ &= \int_0^T \frac{d}{d\varepsilon} F\left(t, s^*(t) + \varepsilon h(t), s^{*'}(t) + \varepsilon h'(t)\right) dt.\end{aligned}$$

Let $x_\varepsilon^* = s^*(t) + \varepsilon h(t)$, $x_\varepsilon^{*'} = s^{*'}(t) + \varepsilon h'(t)$ and $F_\varepsilon = F\left(t, s^*(t) + \varepsilon h(t), s^{*'}(t) + \varepsilon h'(t)\right)$. Then the inside derivative becomes

$$\begin{aligned}\frac{dF_\varepsilon}{d\varepsilon} &= \frac{dt}{d\varepsilon}\frac{\partial F_\varepsilon}{\partial t} + \frac{dx_\varepsilon^*}{d\varepsilon}\frac{\partial F_\varepsilon}{\partial s} + \frac{dx_\varepsilon^{*'}}{d\varepsilon}\frac{\partial F_\varepsilon}{\partial s'} \\ &= h(t)\frac{\partial F_\varepsilon}{\partial s} + h'(t)\frac{\partial F_\varepsilon}{\partial s'}.\end{aligned}$$

Therefore the integral becomes

$$\frac{d}{d\varepsilon}(J(\varepsilon)) = \int_0^T \left[h(t)\frac{\partial F_\varepsilon}{\partial s} + h'(t)\frac{\partial F_\varepsilon}{\partial s'}\right] dt.$$

When $\varepsilon = 0$ we have that J is at its maximum since we chose $s^*(t)$ to be the function that maximizes J. Therefore

$$\left.\frac{dF_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^T \left[h(t)\frac{\partial F}{\partial s} + h'(t)\frac{\partial F}{\partial s'}\right] dt = 0.$$

Using integration by parts and the condition that $h(0) = 0 = h(T)$ the above equation can be rewritten as

$$\int_0^T (\frac{\partial F}{\partial s} - \frac{d}{dt}\frac{\partial F}{\partial s'})h(t)\, dt = 0.$$

For the last step in our derivation we use the following well-know fundamental theorem in Calculus of Variations.

Theorem 1. *If $f(t)$ is a continuous function and*

$$\int_0^T f(t)h(t)\, dt = 0$$

for all continuous and differentiable functions $h(t)$ over $[0,T]$ with $h(0) = 0 = h(T)$, then $f(t) = 0$, for all $t \in [0,T]$.

By the above Theorem, we obtain the Euler-Lagrange equation

$$\frac{\partial F}{\partial s}(t, s^*(t), s^{*'}(t)) - \frac{d}{dt}(\frac{\partial F}{\partial s'}(t, s^*(t), s^{*'}(t))) = 0.$$

4 The Necessary Condition and Optimal Solutions

4.1 The Necessary Condition

We can write the performance function as:

$$F(t,s,s') = \ln(E+\rho s - s')e^{-rt}$$

Taking the partial derivative with respect to s, we get the equation

$$F_s = \frac{\rho e^{-rt}}{E+\rho s - s'}$$

Taking the partial derivative with respect to s', we get the equation

$$F_{s'} = \frac{-e^{-rt}}{E+\rho s - s'}$$

Then replacing $s = s(t)$ and $s' = s'(t)$ and differentiating with respect to t, we get

$$\frac{dF_{s'}}{dt} = \frac{re^{-rt}}{E+\rho s(t) - s'(t)} + \frac{e^{-rt}(\rho s'(t) - s''(t))}{(E+\rho s(t) - s'(t))^2}$$

Note that E, ρ, and r assumed to be independent of time. Please see section 2 for the details.

By taking the derivatives and substituting in the Euler-Lagrange equation that corresponds to our problem, we get the following second order differential equation:

$$(-r+\rho)(\rho s(t) + E - s'(t)) - (\rho s'(t) - s''(t)) = 0,$$

with boundary conditions $s(0) = s_0$ and $s(T) = s_T$.

4.2 Numerical Results: Optimal Monthly Savings and Consumption Plan

We collected data from the National Center for Education Statistics (NCES). The NCES data gives us the average salary of classroom teachers in public elementary and secondary schools in the United States from 1980 to 2017. We then took the average salary and divided it by twelve to find the average monthly salary. Because we were interested in safe investment options, we used data from the U.S. Treasury and the International Monetary Fund to obtain the monthly treasury bond rates which behave similarly to the high yield saving accounts like High Yield American Express account. Interest rates were obtained for 180 months for years $2003-2017$. We also obtained the corresponding monthly U.S. treasury constant maturity (yield-to-maturity) rates data, which was used as a risk-free discounting rate for our numerical simulation.

The descriptive statistics shown by Table 1 provide information about our parameters: salary, treasury yield-to-maturity rate, and treasury 1-year bond yield rate. On average, a teacher makes about $\$53,391$ per year. The minimum starting salary was recorded in

Table 1: Descriptive Statistics

Variable	Observations	Mean	SD	Min	Max
Salary	180	53391	4371.6	45757	58875
Treasury yield-to-maturity	180	0.032	0.0105	0.0150	0.0511
Treasury bond yield	180	0.0144	0.0162	0.0010	0.0522

2003 at $\$45,757$ per year and increased by approximately \$874 every year, reaching the maximum salary of $\$58,875$ per year in 2017. The average treasury yield-to-maturity rates recorded between the years 2003 to 2017 was approximately 3.20%, with a low of 1.5%, which occurred in 2016 and a high of 5.11% between the years 2006 and 2007. The average treasury bond yield rates was around 1.44%, with a minimum of 0.1% from the years 2011 and 2014 and a maximum of 5.22% in 2006.

We want to find our adjusted monthly savings that allows us to continue maximizing the overall satisfaction and meeting the $\$60,000$ target assuming that the teacher continues earning their monthly salary for 15 years. To achieve this goal, we follow the following dynamic programming or scheduling steps.

1. We solve the second order differential equation

$$(-r+\rho)(\rho s(t)+E-s'(t))-(\rho s'(t)-s''(t))=0,$$

with boundary conditions, $s(0)=0$ and $s(15)=60000$. The solution depends on t, ρ, r and E. Let $s(t,\rho,r,E)$ be the solution. Substituting the constant monthly salary for the first year, E_1 into $s_1(t,\rho,r,E)$, we get $s_1(t,\rho,r,E_1)$.
Assume that the monthly treasury bond rates and treasury yield-to-maturity rates are forcasted in advance for the first year (it can also be done quarterly or biannually). Let the first year average treasury bond and treasury yield-to-maturity rates be ρ_1 and r_1. Then the projected monthly savings at the i^{th} month of the first year is $s_1(\frac{i}{12},\rho_1,r_1,E_1)$ for $i=1..12$. These predicted values can be adjusted month-by-month using the first order Taylor's series expansion of $s_1(t,\rho,r,E_1)$:

$$s_1(t_i,\rho_i,r_i,E_1)\cong s_1(t_i,\rho_1,r_1,E_1)+\frac{\partial s_1}{\partial\rho}(t_i,\rho_1,r_1,E_1)(\rho_i-\rho_1)+ \tag{1}$$

$$\frac{\partial s_1}{\partial r}(t_i,\rho_1,r_1,E_1)(r_i-r_1) \tag{2}$$

The first term is the monthly savings predicted using the averages, the second term $\frac{\partial s_1}{\partial\rho}(t_i,\rho_1,r_1,E_1)(\rho_i-\rho_1)$ is the monthly adjustment due to the relative change in treasury bond return rates and the third term $\frac{\partial s_1}{\partial r}(t_i,\rho_1,r_1,E_1)(r_i-r_1)$ is the monthly adjustment due to the relative change in treasury yield-to-maturity rates. Therefore, the adjusted saving at i^{th} month is the difference of the savings account balance between two consecutive months:

$$s_1(\frac{i}{12},\rho_i,r_i,E_1)-s_1(\frac{i-1}{12},\rho_{i-1},r_{i-1},E_1)$$

and our i^{th} month out-of-pocket monthly savings contribution will be

$$\begin{aligned} S_1(i) &= s_1(\frac{i}{12}, \rho_i, r_i, E_1) \\ &- r_{i-1}\left[\sum_{m=1}^{i-1}\left(s_1(\frac{m}{12}, \rho_m, r_m, E_1) - s_1(\frac{m-1}{12}, \rho_{m-1}, r_{m-1}, E_1)\right)\right] \end{aligned} \quad (3)$$

The first year total savings account balance becomes

$$B_1 = \sum_{i=1}^{i=12}\left[s_1(\frac{i}{12}, \rho_i, r_i, E_1) - s_1(\frac{i-1}{12}, \rho_{i-1}, r_{i-1}, E_1)\right],$$

by assumption $s_1(0, \rho_0, r_0, E_1) = 0$.

2. After we finished the first year, month-by-month calculations and got B_1, we solve the same second order differential equation with different boundary conditions, $s(1) = B_1$ and $S(15) = 60000$. Let $s_2(t, \rho, r, E)$ be the solution for this boundary value problem. Again substituting the second year constant monthly salary E_2 in $s_2(t, \rho, r, E)$, we get $s_2(t, \rho, r, E_2)$. Then we repeat step (1) using $s_2(t, \rho_2, r_2, E_2)$ for $i = 13..24$, where ρ_2 and r_2 are the average values of the monthly treasury bond yield and treasury yield-to-maturity rates in year 2. We continue this process recursively for the remaining 13 years.

   We evaluated the monthly savings, monthly consumptions, and cumulative savings account balance for the years $2003 - 2017$ and presented the results in Figure (1), (2) and Figure (3) below.

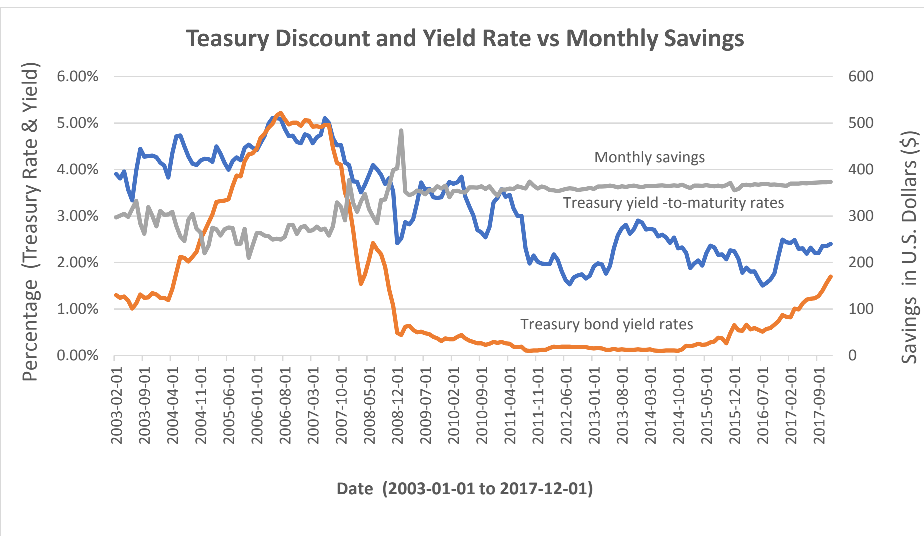

Figure 1: Monthly Adjusted Savings

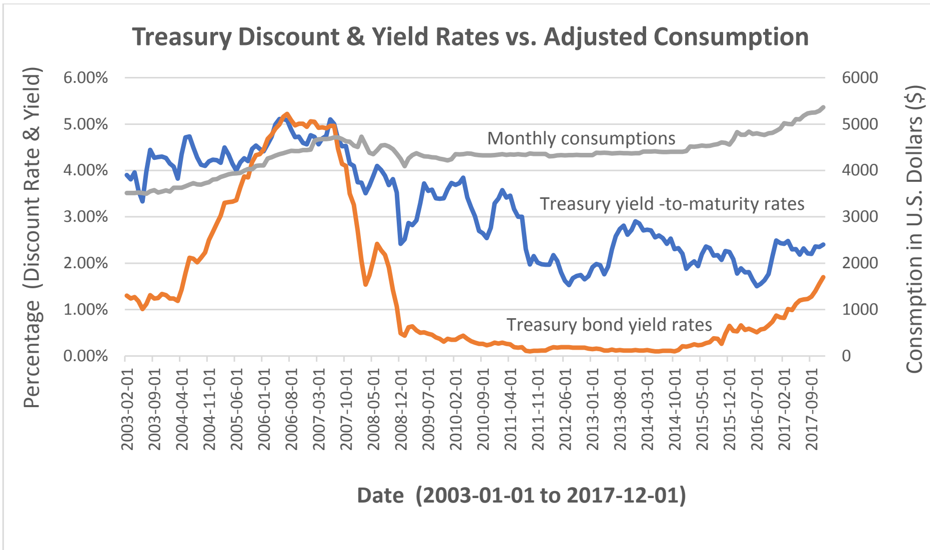

Figure 2: Monthly Adjusted Consumption

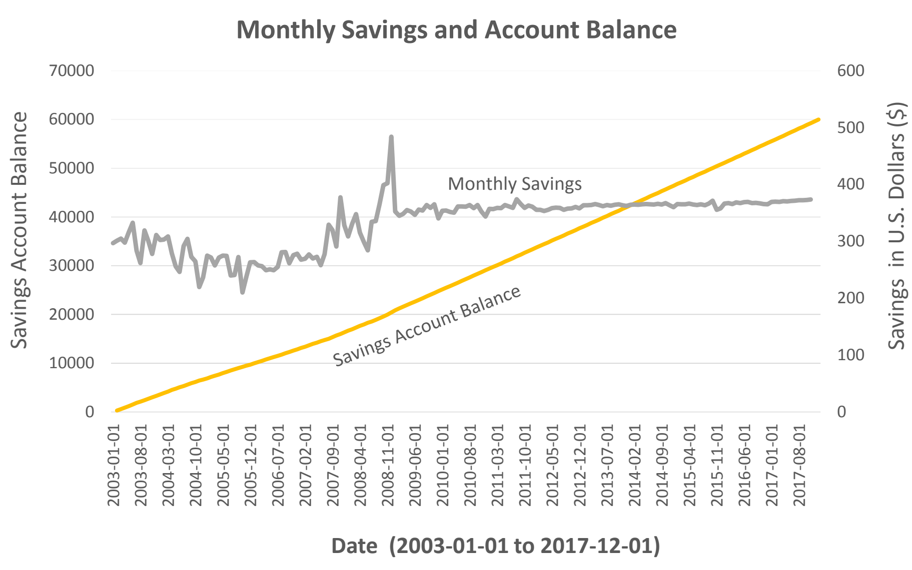

Figure 3: Monthly Savings and Savings Account Balance

Figure (1) and Figure (2) shows adjusted monthly saving and corresponding consumption that will allow the teacher to reach their savings goal of $\$60,000$ in 15 years (please see Figure (3)). From Figure (1) and Figure (2), between the years 2004 to 2007, the teacher would have been able to consume more and save less from their salary. This is because the treasury bond yield rate increased from 1.91% to 5.22%. This allowed the teacher to reach their monthly savings goal by using more of what was coming from the return rather than their salary. However, when the stock market crashed in 2008 and treasury yield rates fell, the teacher would have had to save more and consume less to make up for the low return rate they were getting on their savings account shown by the very low yield rate and keep on track with their goal. When the market began to steady around 2010, the teacher stayed on a fairly constant track with their monthly savings to make sure they reached their goal in the time frame. They were able to consume more towards the end of the fifteen years due to the return increasing again as well as the constant increase in salary per year while getting closer to and eventually reaching their goal of $60,000 in 15 years. It seems as though treasury bond yield rates have a fairly high inverse correlation with adjusted savings, but a very low correlation with adjusted consumption. This makes sense because adjusted savings is based on the fluctuations in the treasury yield-to-maturity rate and treasury bond yield while attempting to reach the $60,000 goal at the end of year 15. On the other hand, adjusted consumption (though impacted by the treasury yield-to-maturity rates and treasury bond yield rates) is more heavily dependent on changes in salary.

5 Conclusion

From our paper, we were able to observe and study how Optimal Control Theory and Dynamic Programming can be applied in fields of economics and finance. To build a savings fund with minimum stress while maximizing our utility of consumption we have used mathematical tools to derive the Euler-Lagrange equation and solve the equation to determine the optimal solution. Later, we solved our problem and considered the complexity of our equations (higher-order differential equations). To complete our numerical and sensitivity analysis and to find adjusted savings and consumption, we have obtained and used the data of teacher's salary, treasury yield-to-maturity rates, and treasury bond yield rates. In the end, we have successfully reached our goal of saving $60,000 at the end of fifteen years of simulation.

What makes our project so essential is the applicability of it in real life. Even though we were assuming the past data as current data, we were able to demonstrate how advanced mathematical computations can be used to work with it. With that being said the same calculations can be used on predicted data rather than the data obtained from the past observations. We hope that our project will bring some insight about applications of Optimal Control Theory in an undergraduate program in the field of Applied Mathematics, Economics,and Finance.

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