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Numerical Range of Strictly Triangular Matrices over Finite Fields

*Ariel Russell**

Ariel Russell produced this work during her time as an undergraduate student at Taylor University. She graduated in 2021 with a B.S. in Mathematics and a B.A. in Computer Science. She now enjoys teaching high school math and programming in Colorado Springs, CO.

Abstract

In this paper we investigate the numerical range of 3×3 matrices over finite fields, particularly when the matrix is strictly triangular. We provide a conjecture for this case that extends to $n \times n$ matrices for $n \geq 3$ and also provide sample code for generating the numerical range.

1 Introduction

Numerical ranges of matrices over $\mathbb{C}$ have been studied extensively, most notably by Hausdorff, Toeplitz, and Kippenhahn. Investigation into numerical ranges over finite fields was initiated in [3] and has been continued in several papers of Ballico (see e.g. [1], [2]). Here we require our field to be of characteristic p where $p \equiv 3 \mod 4$, ensuring that the element -1 is not a quadratic residue in $\mathbb{Z}_p$, so that i has a proper analog to its use in $\mathbb{C}$.

Let p be a prime congruent to $3 \mod 4$, and define $\mathbb{Z}_p[i]$ as the Galois Field of order p^2 in the form $\{a+bi : a,b \in \mathbb{Z}\}$. $M_n(\mathbb{Z}_p[i])$ denotes the set of $n \times n$ matrices with entries in $\mathbb{Z}_p[i]$. The numerical range of matrix $M \in M_n(\mathbb{Z}_p[i])$ is defined as $W(M) = \{x^*Mx : x \in \mathbb{Z}_p[i]^n, \|x\|^2 = x^*x = 1\}$ with x^* representing the conjugate transpose of x. Thus, $W(M)$ forms a set in $\mathbb{Z}_p[i]$. The authors in [3] also introduce the concept of the k-th numerical range, $W_k(M) = \{x^*Mx : x \in \mathbb{Z}_p[i]^n, \|x\|^2 = x^*x = k \in \mathbb{Z}_p\}$. (Here, then, $W(M) = W_1(M)$.)

***Corresponding author**: ariel.j.russell@gmail.com

In [3], work is primarily focused on upper triangular 2×2 matrices. In no 2×2 matrix do we see a numerical range that includes every element of $\mathbb{Z}_p[i]$. It seems one more dimension is needed: in all of our testing, every strictly triangular matrix of dimension 3 or higher had $W(M) = \mathbb{Z}_p[i]$. The goal of this paper is to make as much progress towards that conjecture as possible.

2 Preliminaries

Our proofs in the following sections depend on some key tools. In particular, we frequently attempt to remove one of the entries of an input vector x from the expression x^*Mx, so that the missing entry can ensure that $\|x\|^2 = 1$. The validity of this technique comes from the following two lemmas; the first justifies the second.

Lemma 1. [3, Lemma 2.1] *For all primes p congruent to $3 \mod 4$, and for all $k \in \mathbb{Z}_p$, there exists $t, s \in \mathbb{Z}_p$ for which $t^2 + s^2 = k$.*

Lemma 2. [6, Lemma 5] *Let p be a prime congruent to $3 \mod 4$. For all $k \in \mathbb{Z}_p$ and all $x \in \mathbb{Z}_p[i]$, there exists a $y \in \mathbb{Z}_p[i]$ for which $|x|^2 + |y|^2 \equiv k \mod p$.*

More generally, we will often use unitary equivalence, scaling, and shifting to simplify our calculations. In particular, since for all of our work the resulting numerical range is all of $\mathbb{Z}_p[i]$, any scaling or shifting leaves the result invariant.

Definition 3. [3, Definition 2.5] Let p be a prime congruent to 3 modulo 4 and let $U \in M_n(\mathbb{Z}_p[i])$. We call U a **unitary** matrix if $U^*U = I$.

Lemma 4. [3, Lemma 2.6] *Let $M, U \in M_n(\mathbb{Z}_p[i])$ with U unitary and p a prime congruent to $3 \mod 4$. Then, $W(M) = W(U^*MU)$.*

Lemma 5. [3, Lemma 2.7] *Let p be a prime congruent to $3 \mod 4$ and let $M \in M_n(\mathbb{Z}_p[i])$. For any $a, b \in \mathbb{Z}_p[i]$ we have $W(aM + bI) = aW(M) + b$.*

3 A 0 Entry Above the Diagonal

The following two lemmas appear in an oversimplified form in [6] and in a setting too complex for our needs in [2], and so are reconstructed here. They also have farther-reaching implications than noted in either of those papers, as seen later in this section.

Lemma 6. *For all primes $p \equiv 3 \mod 4$, $W(M) = \mathbb{Z}_p[i]$ where $M \in M_3(\mathbb{Z}_p[i])$ is given by*

$$M = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $a \neq 0$ in $\mathbb{Z}_p[i]$, or any other 3×3 matrix with a single non-zero entry in $\mathbb{Z}_p[i]$ off of the main diagonal.

Proof. First, assume

$$M = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Define $x = (x_1\ x_2\ x_3)^T$, and let $x^*Mx = ax_3\overline{x_1}$ represent elements in the numerical range. (*Note:* $\overline{x_1}$ represents the conjugate of x_1.) For an arbitrary element $k \in \mathbb{Z}_p[i]$, let $x_1 = 1$, and $x_3 = a^{-1}k$, so that $x^*Mx = k$. By Lemma 2, there exists $x_2 \in \mathbb{Z}_p[i]$ such that $|x_2|^2 + |x_3|^2 \equiv 0 \mod p$, so that $|x_1|^2 + |x_2|^2 + |x_3|^2 = 1$. Since k can be any element of $\mathbb{Z}_p[i]$ we have $W(M) = \mathbb{Z}_p[i]$.

If a is in one of the other five spots off of the main diagonal, there is a permutation matrix P so that P^*MP has a in the top-right corner. Since permutation matrices are unitary, by Lemma 4, we still have $W(M) = \mathbb{Z}_p[i]$. □

Lemma 7. *For all primes $p \equiv 3 \mod 4$, $W(M) = \mathbb{Z}_p[i]$ where $M \in M_3(\mathbb{Z}_p[i])$ is given by*

$$M = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$

where $a, c \neq 0$, or any other 3×3 matrix with exactly two non-zero entries, either both above the main diagonal, or both below the main diagonal.

Proof. First assume

$$M = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

Consider $x^*Mx = ax_2\overline{x_1} + cx_3\overline{x_2}$. We will again consider a subset of the numerical range by stipulating that $x_2 = 1$.

First, we show that there is a non-zero element in this set. Letting $x_3 = c^{-1}$, we have that $|x_1|^2 \equiv -|c^{-1}|^2$. By Lemma 1, there exists $A, B \in \mathbb{Z}_p$ such that $A^2 + B^2 \equiv -|c^{-1}|^2$, so we will let $x_1 = A + Bi$. Then $x^*Mx = a(A - Bi) + 1$. This is only 0 if $-a^{-1} = (A - Bi)$, in which case we can instead begin by choosing $x_3 = -c^{-1}$, and use the same choice for x_1.

Now, let $a\overline{x_1} + cx_3$ be a fixed non-zero quantity with the constraint that $|x_1|^2 + |x_3|^2 \equiv 0$. Let us now consider $\overline{k}x_1$ and kx_3 where k is an arbitrary element of $\mathbb{Z}_p[i]$. Note that $|\overline{k}x_1|^2 + |kx_3|^2 = |k|^2|x_1|^2 + |k|^2|x_3|^2 = |k|^2(|x_1|^2 + |x_3|^2) = |k|^2(0) = 0$, which satisfies the constraint. Then, the output becomes $ak\overline{x_1} + kcx_3 = k(a\overline{x_1} + cx_3)$. Since $a\overline{x_1} + cx_3$ is fixed and k varies over all of $\mathbb{Z}_p[i]$, we have that $k(a\overline{x_1} + cx_3)$ maps to every element of $\mathbb{Z}_p[i]$, since $k \to \overline{a}^{-1}k$ is an automorphism of $\mathbb{Z}_p[i]$ (where $\overline{a}^{-1} = a\overline{x_1} + cx_3 \in \mathbb{Z}_p[i]^*$). Therefore, $W(M) = \mathbb{Z}_p[i]$.

Now, if M has its two non-zero elements in other entries off of the main diagonal, the roles of x_1, x_2, x_3 can be adjusted accordingly to achieve the same result. For example, if $M = \begin{pmatrix} 0 & a & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, let $x_1 = 1$ and apply the same argument to $ax_2 + cx_3$. Similarly, if $M = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$, let $x_3 = 1$ and apply the argument to $a\overline{x_1} + c\overline{x_2}$. If M is instead strictly lower triangular, then conjugation by the standard exchange matrix (which is unitary) will change it to a strictly upper triangular matrix, while preserving the numerical range, so that the prior results may be applied. □

In [2], Lemmas 6 and 7 are considered only as 3×3 without considering higher dimensions, and in [6] little variance is given to where the entries appear and with what values, although higher dimensions are considered. However by using submatrices, these results extend to matrices of arbitrary size.

Lemma 8. *Suppose M_i is a principal submatrix of M created by deleting the ith row and ith column of M. Then $W(M_i) \subseteq W(M)$.*

Proof. If x' is generated from $x \in \mathbb{Z}_p[i]^{n-1}$ by inserting a 0 in position i, then $\|x'\| = \|x\|$ and $\langle x', Mx' \rangle = \langle x, M_i x \rangle$. □

Theorem 9. *Suppose $M \neq 0$ is an $n \times n$ triangular matrix with elements in $\mathbb{Z}_p[i]$ and $n \geq 3$, and a constant diagonal. Suppose also that at least one element above the main diagonal (if M is upper triangular) or below the main diagonal (if M is lower triangular) is 0. Then $W(M) = \mathbb{Z}_p[i]$.*

Proof. If M is strictly lower triangular, it is unitarily equivalent by a permutation matrix (the standard exchange matrix) to a strictly upper triangular matrix, so we will assume M is strictly upper triangular without loss of generality.

We will proceed by induction on n. For the base case ($n = 3$), the statement is a direct corollary of Lemmas 6 and 7.

Suppose $n \geq 4$, and assume for any strictly upper triangular, non-zero $(n-1) \times (n-1)$ matrix with at least one 0 above the main diagonal and all entries in $\mathbb{Z}_p[i]$, the numerical range is $\mathbb{Z}_p[i]$.

If the constant diagonal is not 0, then we may use Lemma 5 and achieve the same result.

For an $n \times n$ matrix M with the same hypotheses, consider the principal submatrix M_i by deleting a row and corresponding column which does not remove all of the zeroes above the diagonal. If M_i is the zero matrix, since $n \geq 4$, there are other rows and corresponding columns that can be instead deleted so that M_i is not 0, while keeping at least one 0 above the diagonal. Once M_i is correctly chosen, by our inductive hypothesis, $W(M_i) = \mathbb{Z}_p[i]$. Then by Lemma 8, $\mathbb{Z}_p[i] = W(M_i) \subseteq W(M)$, so $W(M) = \mathbb{Z}_p[i]$. □

There is a clear third case missing: what if all three entries above the diagonal in a 3×3 matrix are non-zero? Unfortunately, this problem has proved particularly vexing. Testing indicates that all strictly triangular matrices M have $W(M) = \mathbb{Z}_p[i]$, but we are unable to resolve the last piece of the puzzle. In the next section, we will achieve some results in this situation for 4×4 matrices and higher.

It is also worth noting that we are considering *strictly* triangular matrices for another reason beyond simplicity of calculations. In, [3, Example 4.1] a block-reduced upper triangular 3×3 matrix is shown with $W(M)\neq \mathbb{Z}_p[i]$.However, while maintaining some 0 entries above the diagonal, we are able to allow some more variance along the diagonal. This next result can be conjugated by permutations to give results for other similar 3×3 matrices, but this specific form will be useful in the next section, so we leave it as is.

Conjecture 3.5 Suppose M is a 3×3 matrix of the form

$$M=\begin{pmatrix} a & b & 0\\ 0 & a & 0\\ 0 & 0 & 0\end{pmatrix}$$

with all elements in $\mathbb{Z}_p[i]$, $b\neq 0$. Then $W(M)=\mathbb{Z}_p[i]$.

Though we have not completed a proof for this conjecture, we believe such a proof would be possible. As we will show in the next section, the consequences of this conjecture could expand our results to new dimensions.

4 No 0 Entries Above the Diagonal

Here we are able to make progress on some strictly triangular 4×4 matrices with no 0 entries above the diagonal, and then generalize to higher dimensions. The work depends on results about 2×2 matrices.

Theorem 10. *Suppose*

$$M=\begin{pmatrix} 0 & a & b & c\\ 0 & 0 & d & e\\ 0 & 0 & 0 & f\\ 0 & 0 & 0 & 0\end{pmatrix}$$

with $a,b,c,d,e,f\neq 0$ belonging to $\mathbb{Z}_p[i]$. Suppose further that the 2×2 matrix

$$T=\begin{pmatrix} -\overline{a}^{-1}\overline{b}d & (f-\overline{a}^{-1}\overline{b}e)\\ -\overline{a}^{-1}\overline{c}d & -\overline{a}^{-1}\overline{c}e\end{pmatrix}$$

is such that any element of $\mathbb{Z}_p[i]$ can be represented as y^*Ty where $y\in\mathbb{Z}_p[i]^2$. Then $W(M)=\mathbb{Z}_p[i]$.

Proof. Keep in mind that if $x=(x_1\ x_2\ x_3\ x_4)^T$ is a vector with entries in $\mathbb{Z}_p[i]$, then a typical numerical range element looks like:

$$x^*Mx=\overline{x_1}(ax_2+bx_3+cx_4)+\overline{x_2}(dx_3+ex_4)+f\overline{x_3}x_4.$$

Since $a\neq 0$, it is invertible, we can let $x_2=-a^{-1}(bx_3+cx_4)$. Then, the expression becomes

$$\overline{x_1}(ax_2+bx_3+cx_4)+\overline{x_2}(dx_3+ex_4)+f\overline{x_3}x_4= \tag{1}$$

$$0-\overline{a}^{-1}(\overline{bx_3}+\overline{cx_4})(dx_3+ex_4)+f\overline{x_3}x_4= \tag{2}$$

$$-\overline{a}^{-1}\overline{b}d|x_3|^2-\overline{a}^{-1}\overline{c}e|x_4|^2+(f-\overline{a}^{-1}\overline{b}e)\overline{x_3}x_4-\overline{a}^{-1}\overline{c}d\overline{x_4}x_3. \tag{3}$$

And this final expression represents y^*Ty if $y = (x_3\ x_4)^T$ and

$$T = \begin{pmatrix} -\overline{a}^{-1}\overline{b}d & (f - \overline{a}^{-1}\overline{b}e) \\ -\overline{a}^{-1}\overline{c}d & -\overline{a}^{-1}\overline{c}e \end{pmatrix}.$$

Note that while we are assuming x_2 has a specific form, we have made no assumptions about x_1, x_3, x_4. We can let x_3, x_4 be any values in $\mathbb{Z}_p[i]$, and by Lemma 2, x_1 can always be chosen so that $|x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2 = 1$. Since we assume that y^*Ty can represent any element of $\mathbb{Z}_p[i]$ when x_3 and x_4 can be freely chosen, we are done. □

The assumption of representation in the Theorem 10 is equivalent to requiring that the 2×2 matrix T satisfies $\bigcup_{k\in\mathbb{Z}_p} W_k(T) = \mathbb{Z}_p[i]$. Prior work in [6] and [3] help answer this question.

Lemma 11 ([6, Lemma 10])**.** *Let $A \in M_n(\mathbb{Z}_p[i])$ and let B be the block matrix $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$. Then $W(B) = \bigcup_{k\in\mathbb{Z}_p} W_k(A)$.*

In [3]; 2×2 numerical ranges are largely reduced to a few specific cases; we will consider those as parts of 3×3 block matrices in the following proof.

Proposition 12 (Corollary of Conjecture 3.5)**.** *Suppose $M \in M_2(\mathbb{Z}_p[i])$ has a single (repeated) eigenvalue in $\mathbb{Z}_p[i]$, with corresponding eigenvectors $v \in \mathbb{Z}_p[i]^2$ satisfying $\|v\|^2 \neq 0$, and M is irreducible. Then $\bigcup_{k\in\mathbb{Z}_p} W_k(M) = \mathbb{Z}_p[i]$.*

Proof. By Lemma 11, we need only show that for any such M, $W(B) = \mathbb{Z}_p[i]$ where $B = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$. By [3, Theorem 1.2], M is unitarily equivalent to an upper triangular matrix; since it has a single eigenvalue, we can write $M = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. By Conjecture 3.5 and Lemma 4.2, $W(B) = \bigcup_{k\in\mathbb{Z}_p} W_k(M) = \mathbb{Z}_p[i]$. □

Putting these together, we can see a clearer form of Theorem 10.

Proposition 13 (Corollary of Conjecture 3.5)**.** *Suppose*

$$M = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $a, b, c, d, e, f \neq 0$ belonging to $\mathbb{Z}_p[i]$. Suppose further that the 2×2 matrix

$$T = \begin{pmatrix} -\overline{a}^{-1}\overline{b}d & (f - \overline{a}^{-1}\overline{b}e) \\ -\overline{a}^{-1}\overline{c}d & -\overline{a}^{-1}\overline{c}e \end{pmatrix}$$

has a single (repeated) eigenvalue in $\mathbb{Z}_p[i]$, with all eigenvectors v satisfying $\|v\|^2 \neq 0$, and T is irreducible. Then $W(M) = \mathbb{Z}_p[i]$.

Proof. The proof follows immediately from Theorem 10 and Proposition 12. □

Corollary 14. *Suppose M is an $n \times n$ matrix, $n \geq 4$, with a constant diagonal and all entries above the diagonal constant (possibly different from the diagonal). Then $W(M) = \mathbb{Z}_p[i]$.*

Proof. If $n \geq 4$, consider a 4×4 submatrix M' of M. By Lemma 5, we need only consider

$$M' = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding 2×2 matrix in Proposition 13 is $T = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$. This matrix has a single, repeated eigenvalue in $\mathbb{Z}_p[i]$, with eigenvector $v = (0\ 1)^T$ satisfying $\|v\|^2 \neq 0$, so the result follows from Proposition 13. $\square$

Of course, having results for 4×4 then generalizes to higher dimensions.

Corollary 15. *Suppose that $M \in M_n(\mathbb{Z}_p[i]), n \geq 5$, has a principal submatrix that satisfies the conditions of Proposition 13. Then $W(M) = \mathbb{Z}_p[i]$.*

Proof. This follows directly from Theorem 10 and Proposition 12. $\square$

Unfortunately, though Theorem 10 and Proposition 12 are sufficient to prove Corollary 15, they are not necessary. The matrix

$$M = \begin{pmatrix} 0 & 1 & 4+2i & 4+4i \\ 0 & 0 & 1+6i & 1+6i \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

over $\mathbb{Z}_7[i]$ has a full numerical range, but the corresponding 2×2 matrix

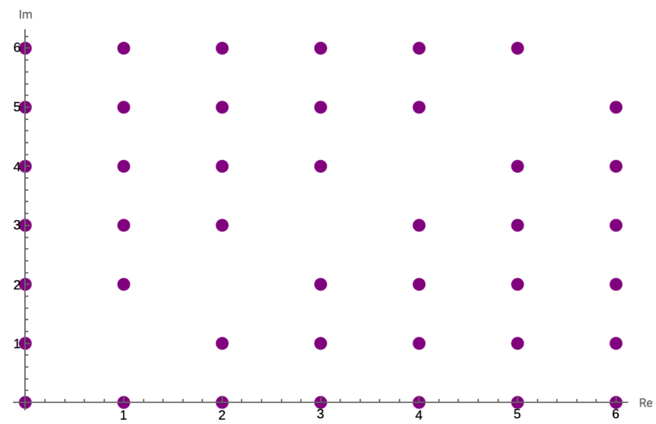
$$T = \begin{pmatrix} 5+6i & 6i \\ i & i \end{pmatrix}$$

does not have $\bigcup_{k \in \mathbb{Z}_7} W_k(T) = \mathbb{Z}_7[i]$. By Lemma 11, this can be seen by viewing $W(B)$ where

$$B = \begin{pmatrix} 5+6i & 6i & 0 \\ i & i & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

That image is shown in Figure 1. It is noteworthy that the eigenvalues of T, $\frac{1}{2}((5+7i) \pm \sqrt{-24+50i})$, do not belong to $\mathbb{Z}_7[i]$.

Figure 1: $W(M) = \mathbb{Z}_7[i]$, but $\bigcup_{k\in\mathbb{Z}_7} W_k(T) \neq \mathbb{Z}_7[i]$.



5 Future Work

Our biggest concern is finishing the case of 3×3 strictly triangular matrices. We feel fairly confident in the following conjecture.

Conjecture. *If $M \in M_n(\mathbb{Z}_p[i]), n \geq 3$ is strictly triangular, then $W(M) = \mathbb{Z}_p[i]$.*

More broadly, we believe, based on our preliminary explorations, that the numerical range of all 3×3 matrices over $\mathbb{Z}_p[i]$ can be classified into one of a few finite categories. Much work still needs done to identify the criteria for determining the size of the numerical range of a given matrix, but examples of each of these numerical range shapes may be found in Appendix .

Conjecture. *If $M \in M_3(\mathbb{Z}_p[i])$, then $W(M)$ contains either 1 element, p elements, p^2-1 elements, $p^2-(p-1)$ elements, or p^2 elements.*

Beyond that, in [3, Propositon 3.4], a variation of Schur's Theorem for 2×2 matrices over $\mathbb{Z}_p[i]$ is established. If that generalizes to higher dimensions, then we have the following conjecture.

Conjecture. *If $M \in M_n(\mathbb{Z}_p[i]), n \geq 3$ has a single eigenvalue, belonging to $\mathbb{Z}_p[i]$, with all eigenvectors v satisfying $\|v\|^2 \neq 0$, then $W(M) = \mathbb{Z}_p[i]$.*

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Appendices

1 Examples of Various Sizes of Numerical Ranges

1.1 $W(M) = \mathbb{Z}$

As established, we have reason to believe that strictly triangular matrices in $\mathbb{Z}$ have $W(M) = \mathbb{Z}$. However, we can see by these examples that such matrices are not the *only* matrices to satisfy this.

Consider the following examples of matrices $M \in \mathbf{Z}_7[i]$ which satisfy $W(M) = \mathbb{Z}$.

As in Figure 2, each of the following matrices $M \in \mathbf{Z}_7[i]$ satisfies $W(M) = \mathbb{Z}$.

- $M = \begin{pmatrix} 1+4i & 5i & 4+5i \\ 1+2i & 2i & 6+2i \\ 2+i & 3+5i & 1+6i \end{pmatrix}$
- $M = \begin{pmatrix} 1+i & 2+6i & 6+i \\ 2+3i & 5+6i & 3+3i \\ 5+i & 4 & i \end{pmatrix}$
- $M = \begin{pmatrix} 4+6i & 3 & 1+2i \\ 2+i & 4+6i & 1+3i \\ 2+4i & 2+4i & 4i \end{pmatrix}$
- $M = \begin{pmatrix} 4+6i & 1+3i & 4+2i \\ 4+5i & 5+4i & 0 \\ 3 & 6+i & 3+6i \end{pmatrix}$
- $M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$
- $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

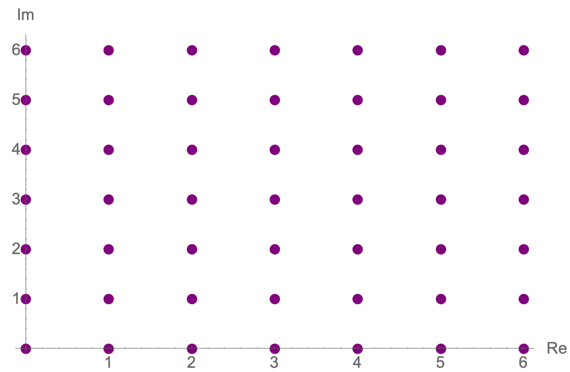

Figure 2: The numerical range of M when $W(M) = \mathbb{Z}_7[i]$

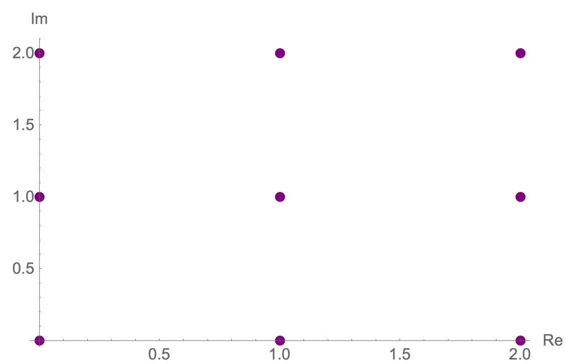

Figure 3: The numerical range of M when $W(M) = \mathbb{Z}_3[i]$

Below, we also have a selection of examples of $M \in \mathbf{Z}_3[i]$ which satisfy $W(M) = \mathbf{Z}_3[i]$, as in Figure 3

- $$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$
- $$M = \begin{pmatrix} 1+i & 2 & 2i \\ 1 & 0 & 1 \\ 1 & 0 & 1+i \end{pmatrix}$$
- $$M = \begin{pmatrix} 1+i & 2 & 2i \\ 1 & i & i \\ 2 & 2i & 1+i \end{pmatrix}$$
- $$M = \begin{pmatrix} 1 & i & 1 \\ i & 1 & i \\ 1 & i & 1 \end{pmatrix}$$

1.2 $|W(M)| = p^2 - 1$

We have no current conjecture regarding how to classify these matrices, but in our exploration we identified several instances of where $W(M)$ contains all but one element of $\mathbb{Z}$. A selection of examples are given below.

Example. $M \in \mathbf{Z}_7[i], M = \begin{pmatrix} 0 & 6+5i & 3i \\ 4+3i & 0 & 6+5i \\ 2+5i & 4+3i & 0 \end{pmatrix}$ with $W(M)$ as shown in Figure

4.

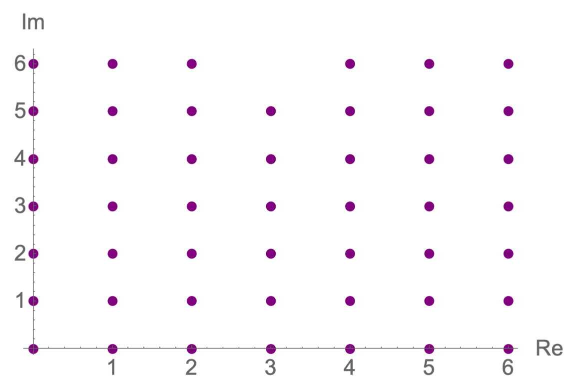

Figure 4: $W(M)$ with $M = \begin{pmatrix} 0 & 6+5i & 3i \\ 4+3i & 0 & 6+5i \\ 2+5i & 4+3i & 0 \end{pmatrix}$ over $\mathbf{Z}_7[i]$

Example. $M \in \mathbf{Z}_7[i], M = \begin{pmatrix} 4+i & 5+3i & 5i \\ 2+3i & 1+4i & 4i \\ 1+i & 4i & 4+3i \end{pmatrix}$ with $W(M)$ as shown in Figure 5.

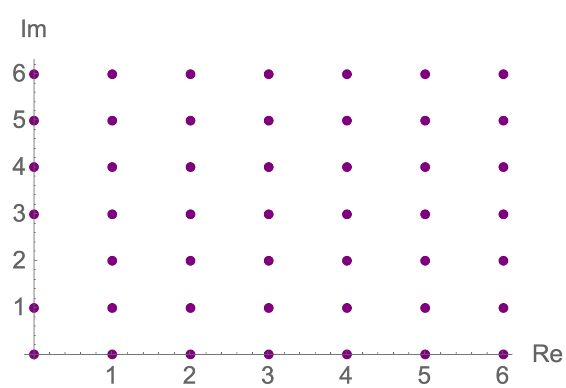

Figure 5: $W(M)$ with $M = \begin{pmatrix} 4+i & 5+3i & 5i \\ 2+3i & 1+4i & 4i \\ 1+i & 4i & 4+3i \end{pmatrix}$ over $\mathbf{Z}_7[i]$

Example. $M \in \mathbf{Z}_3[i], M = \begin{pmatrix} 2+2i & 2i & 2i \\ 1+2i & 0 & 2+2i \\ 1 & 2 & 1+2i \end{pmatrix}$ with $W(M)$ as shown in Figure 6.

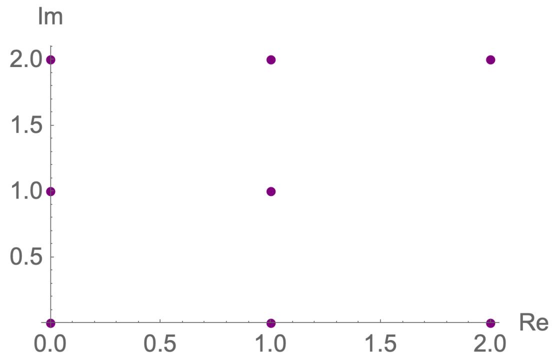

Figure 6: $W(M)$ with $M = \begin{pmatrix} 2+2i & 2i & 2i \\ 1+2i & 0 & 2+2i \\ 1 & 2 & 1+2i \end{pmatrix}$ over $\mathbf{Z}_3[i]$

Example. $M \in \mathbf{Z}_3[i], M = \begin{pmatrix} 1 & 2+i & i \\ 2 & 0 & 1+i \\ 2+2i & i & 0 \end{pmatrix}$ with $W(M)$ as shown in Figure 7.

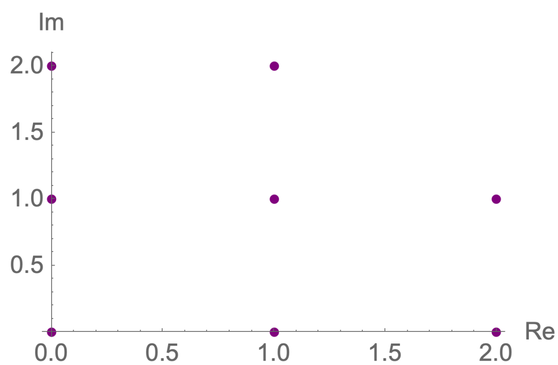

Figure 7: $W(M)$ with $M = \begin{pmatrix} 1 & 2+i & i \\ 2 & 0 & 1+i \\ 2+2i & i & 0 \end{pmatrix}$ over $\mathbf{Z}_3[i]$

1.3 $|W(M)| = p^2 - (p-1)$

We also provide here a selection of examples where $W(M)$ is missing $p-1$ elements of $\mathbb{Z}$. Intuitively, this means that there is nearly a "line" missing.

Example. $M \in \mathbf{Z}_7[i], M = \begin{pmatrix} 0 & 3 & 5 \\ 6 & 0 & 3 \\ 4 & 6 & 0 \end{pmatrix}$ with $W(M)$ as shown in Figure 8. You can see that all elements $6+bi, b \neq 0$ are excluded from the numerical range.

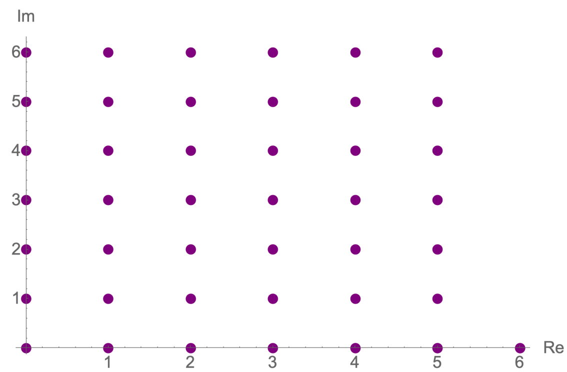

Figure 8: $W(M)$ with $M = \begin{pmatrix} 0 & 3 & 5 \\ 6 & 0 & 3 \\ 4 & 6 & 0 \end{pmatrix}$ over $\mathbf{Z}_7[i]$

Example. $M \in \mathbf{Z}_3[i], M = \begin{pmatrix} 0 & 2+3i & 1+5i \\ 4+6i & 0 & 2+3i \\ 5+4i & 4+6i & 0 \end{pmatrix}$ with $W(M)$ as shown in Figure 9. This matrix is the same as the previous matrix, scaled by a factor of $3+i$. Transformations like this rotate the numerical range according to the factor it was scaled by. In Figure 9, the "missing line" is still visible, identifiable with a "slope" of 4.

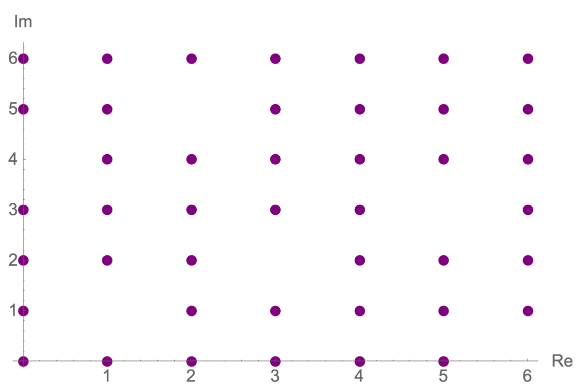

Figure 9: $W(M)$ with $M = \begin{pmatrix} 0 & 2+3i & 1+5i \\ 4+6i & 0 & 2+3i \\ 5+4i & 4+6i & 0 \end{pmatrix}$ over $\mathbf{Z}_7[i]$

Example. $M \in \mathbf{Z}_3[i], M = \begin{pmatrix} 1+i & 2 & i \\ 2+2i & 2i & 0 \\ i & 2+i & 2 \end{pmatrix}$ with $W(M)$ as shown in Figure 10. We see that the elements $1+2i$ and $2+i$ are not included in the numerical range.

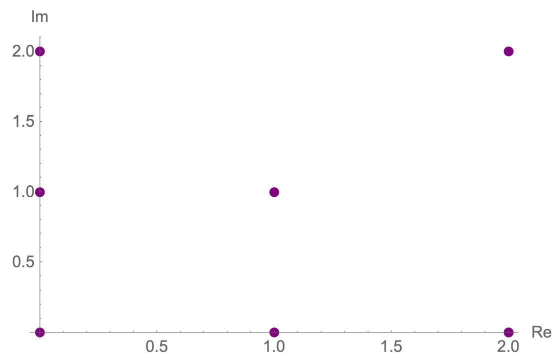

Figure 10: $W(M)$ with $M = \begin{pmatrix} 1+i & 2 & i \\ 2+2i & 2i & 0 \\ i & 2+i & 2 \end{pmatrix}$ over $\mathbf{Z}_3[i]$

1.4 $|W(M)| = p$

Matrices which are equal to their own conjugate transpose have a numerical range of $\mathbf{Z}_p$. Multiples of such matrices have numerical ranges with p elements, rotated off of the $\mathbf{Z}_p$ line according the the factor the matrix was scaled by.

Example. $M \in \mathbf{Z}_7[i], M = \begin{pmatrix} 0 & 5i & 3i \\ 2i & 0 & 5i \\ 4i & 2i & 0 \end{pmatrix}$ with $W(M)$ as shown in Figure 11.

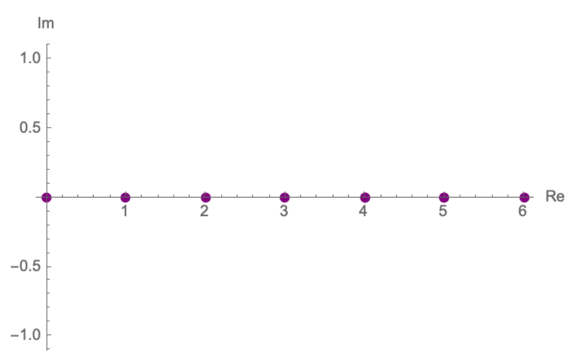

Figure 11: $W(M)$ with $M = \begin{pmatrix} 0 & 5i & 3i \\ 2i & 0 & 5i \\ 4i & 2i & 0 \end{pmatrix}$ over $\mathbf{Z}_7[i]$

Example. $M \in \mathbf{Z}_7[i], M = \begin{pmatrix} 0 & 2+5i & 4+3i \\ 5+2i & 0 & 2+5i \\ 3+4i & 5+2i & 0 \end{pmatrix}$ with $W(M)$ as shown in Figure 12. This matrix is the previous example, scaled by a factor of $1+i$.

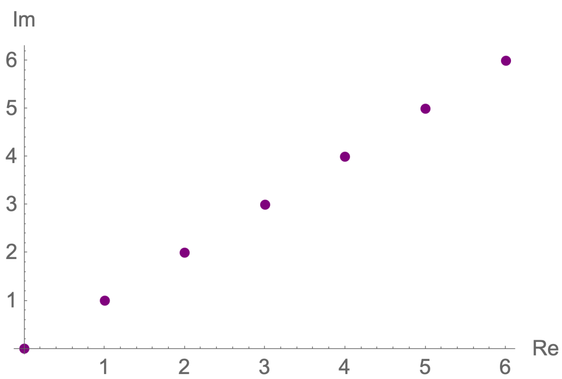

Figure 12: $W(M)$ with $M = \begin{pmatrix} 0 & 2+5i & 4+3i \\ 5+2i & 0 & 2+5i \\ 3+4i & 5+2i & 0 \end{pmatrix}$ over $\mathbf{Z}_7[i]$

1.5 $|W(M)| = 1$

The zero matrix (or a shifted zero matrix) has only 1 element in its numerical range.

Example. $M \in \mathbf{Z}_3[i], M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with $W(M)$ as shown in Figure 13.

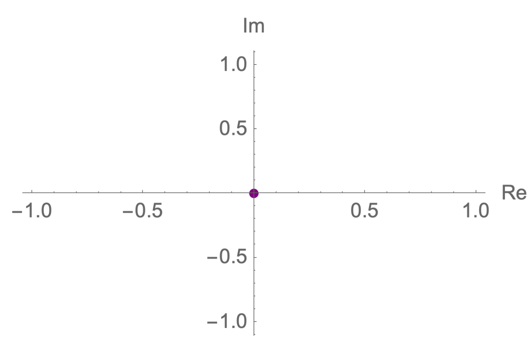

Figure 13: $W(M)$ with $M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ over $\mathbf{Z}_3[i]$

2 Mathematica Code to Replicate Results

In our research, we relied heavily on computation to explore and verify results. We include key aspects of our Mathematica code here for the purposes of replication. Great thanks to Amish Mishra for writing the original version of the code, which we have adapted to be what is included here.

2.1 Preliminaries

Several functions and variables must be defined in order to calculate and plot the numerical range of a matrix over a finite field. Since the built-in functions do not account for finite fields, we must build our own.

```
p = 3; (* Change for different size finite field *)

plotNumericalRange[numRange_] := (
plotPoints = {};
Do[
AppendTo[
plotPoints, {Re[numRange[[each]][[1]][[1]]],
        Im[numRange[[each]][[1]][[1]]]}];
, {each, 1, Length[numRange]}];
ListPlot[plotPoints,
PlotStyle -> Directive[Purple, PointSize[.02]],
AxesLabel -> {Re, Im}]
)

zpi = {};
Do[
Do [
AppendTo[zpi, a + b*I];
, {b, 0, p - 1}];
, {a, 0, p - 1}];

ZpiArray3x3[p_, n_]  := (
Module[{Z1, num, Zpi},
Z1 = ConstantArray[0, {p, p, p}]; (* TODO: make this dynamic,
maybe with another ConstantArray? *)
Do[
Clear[num];
Do[
num = a + b*I;
Z1[[a + 1, b + 1]] = num;
, {b, 0, p - 1}]
, {a, 0 , p - 1}];
Zpi = Tuples[Flatten[Z1], {n}];
Zpi
]
)
numericalRange3x3[k_, M_, p_, n_] := (
Module[{numRange, Zpi},
numRange = {};
Zpi = ZpiArray3x3[p, n];
Do[
x = Zpi[[idx]];
```

```
norm = Mod[x.{x}\[ConjugateTranspose], p][[1]];
If[norm == k,
numRangeElem =
Mod[Transpose[{x}]\[ConjugateTranspose].M.Transpose[{x}], p];
If[! MemberQ[numRange, numRangeElem],
AppendTo[numRange, numRangeElem]
]
];
, {idx, 1, Length[Zpi]}];
(* This portion makes all terms of the numerical range the \
positive modulo p. *)
Do[
If[Re[numRange[[idx]]][[1]][[1]] < 0,
numRange[[idx]][[1]][[1]] = numRange[[idx]][[1]][[1]] + p;
];
If[Im[numRange[[idx]]][[1]][[1]] < 0,
numRange[[idx]][[1]][[1]] = numRange[[idx]][[1]][[1]] + p*I;
];
, {idx, 1, Length[numRange]}];
nr = {};
Do[
If[! MemberQ[nr, numRange[[i]]],
AppendTo[nr, numRange[[i]]];
];
, {i, 1, Length[numRange]}];
nr]
)
```

2.2 Plotting a numerical range

Once we have our functions defined, we can utilize them to calculate and plot a numerical range.

```
(* Change M to a 3x3 matrix here. To get an
imaginary symbol, type esc i i esc *)

M = {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}};

Print["M: ", MatrixForm[M]];
plotNumericalRange[numericalRange3x3[0, M, p, 3]]
```