Another view of the coarse invariant $\sigma$

Takuma Imamura

Takuma Imamura is a Ph.D. student at Kyoto University majoring in mathematical logic. This article is a part of a research project which aims to develop nonstandard methods in large-scale topology.

Abstract Miller, Stibich and Moore [6] developed a set-valued coarse invariant $\sigma(X, \xi)$ of pointed metric spaces. DeLyser, LaBuz and Tobash [2] provided a different way to construct $\sigma(X, \xi)$ (as the set of all sequential ends). This paper provides yet another definition of $\sigma(X, \xi)$. To do this, we introduce a metric on the set $S(X, \xi)$ of coarse maps $(\mathbb{N}, 0) \to (X, \xi)$, and prove that $\sigma(X, \xi)$ is equal to the set of coarsely connected components of $S(X, \xi)$. As a by-product, our reformulation trivialises some known theorems on $\sigma(X, \xi)$, including the functoriality and the coarse invariance.

1 Introduction

Miller, Stibich and Moore [6] developed a set-valued coarse invariant $\sigma(X, \xi)$ of $\sigma$-stable pointed metric spaces $(X, \xi)$. DeLyser, LaBuz and Wetsell [3] generalised it to pointed metric spaces (without $\sigma$-stability). The coarse invariance of $\sigma(X, \xi)$ was proved by Fox, LaBuz and Laskowsky [4] for $\sigma$-stable spaces, and by DeLyser, LaBuz and Wetsell [3] for general spaces.

We start with recalling the definition of $\sigma(X, \xi)$. We adopt a simplified definition given by DeLyser, LaBuz and Tobash [2]. Let $(X, \xi)$ be a pointed metric space. A coarse sequence in $(X, \xi)$ is a coarse map $s : (\mathbb{N}, 0) \to (X, \xi)$. Denote the set of coarse sequences in $(X, \xi)$ by $S(X, \xi)$. Given $s, t \in S(X, \xi)$, we write $s \equiv_{X, \xi}^\sigma t$ if $s$ is a subsequence of $t$. Denote the equivalence closure of $\equiv_{X, \xi}^\sigma$ by $\equiv_{X, \xi}$. In other words, $s \equiv_{X, \xi} t$ if and only if there exists a finite sequence $(u_i)_{i=0}^n$ in $S(X, \xi)$ such that $u_0 = s$, $u_n = t$, and $u_i \equiv_{X, \xi} u_{i+1}$ or $u_{i+1} \equiv_{X, \xi} u_i$ for
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all $i < n$. The desired invariant is then defined as the quotient set:

$$\sigma (X, \xi) := S(X, \xi) / \equiv_{X, \xi}^\sigma$$

$$:= \{ [s]_{X, \xi}^\sigma \mid s \in S(X, \xi) \},$$

where $[s]_{X, \xi}^\sigma$ is the $\equiv_{X, \xi}^\sigma$-equivalence class of $s$. As noted in [5], there is no difficulty in generalising $\sigma (X, \xi)$ to pointed coarse spaces $(X, \xi)$. See the subsection Notation and terminology below for the definitions of the terms used here.

DeLyser, LaBuz and Tobash [2] provided an alternative definition of $\sigma (X, \xi)$. Suppose $(X, \xi)$ is a pointed metric space. Two coarse sequences $s, t \in S(X, \xi)$ are said to converge to the same sequential end (and denoted by $s \equiv_{X, \xi} t$) if there is a $K > 0$ such that for all bounded subsets $B$ of $X$ there is an $N \in \mathbb{N}$ such that $\{s(i) | i \geq N\} \text{ and } \{t(i) | i \geq N\}$ are contained in the same $K$-chain-connected component of $X \setminus B$. The $\equiv_{X, \xi}^\sigma$-equivalence classes are called sequential ends in $(X, \xi)$. It was proved that $\equiv_{X, \xi}^\sigma \text{ and } \equiv_{X, \xi}^\sigma$ coincide. As a result, $\sigma (X, \xi)$ is equal to the set of sequential ends in $(X, \xi)$. This gives another view of $\sigma (X, \xi)$.

This paper aims to provide yet another view of $\sigma (X, \xi)$. Consider the following diagram:

where $\text{Coarse}_*$ is the category of pointed coarse spaces and (base point preserving) coarse maps, $\text{Metr}_b$ the category of metric spaces and bornologous maps, $\text{Coarse}_b$ the category of coarse spaces and bornologous maps, and $\text{Sets}$ the category of sets and maps. In Section 2, we introduce the so-called coarsely connected component functor $Q : \text{Coarse}_b \to \text{Sets}$. The coarse invariance of $Q$ is proved. In Section 3, we introduce a metric on the set $S(X, \xi)$, where the metric is allowed to take the value $\infty$. This forms a functor $S : \text{Coarse}_* \to \text{Metr}_b$. We prove the preservation of bornotopy by $S$. In Section 4, we prove that $\sigma$ can be considered as the composition of the two functors $Q$ and $S$, which commutes the above diagram. As a by-product, our reformulation trivialises some known theorems on $\sigma (X, \xi)$, including the functoriality and the coarse invariance.
Notation and terminology

Let \( f, g : X \to Y \) be maps, \( E, F \) binary relations on \( X \) (i.e. subsets of \( X \times X \)), and \( n \in \mathbb{N} \). Then

\[
E \circ F := \{(x, y) \in X \times X | (x, z) \in E \text{ and } (z, y) \in F \text{ for some } z \in X \},
\]

\[
E^{-1} := \{(y, x) \in X \times X | (x, y) \in E \},
\]

\[
E^0 := \Delta_X := \{(x, x) | x \in X \},
\]

\[
E^{n+1} := E^n \circ E,
\]

\[
(f \times g)(E) := \{(f(x), g(y)) | (x, y) \in E \}.
\]

A coarse structure on a set \( X \) is a family \( C_X \) of binary relations on \( X \) with the following properties:

1. \( \Delta_X \in C_X \);
2. \( E \subseteq F \in C_X \implies E \in C_X \); and
3. \( E, F \in C_X \implies E \cup F, E \circ F, E^{-1} \in C_X \).

A set equipped with a coarse structure is called a coarse space. A subset \( A \) of \( X \) is called a bounded set if \( A \times A \in C_X \). We denote the family of bounded subsets of \( X \) by \( B_X \). This family satisfies the following:

1. \( \bigcup B_X = X \);
2. \( A \subseteq B \in B_X \implies A \in B_X \);
3. \( A, B \in B_X, A \cap B \neq \emptyset \implies A \cup B \in B_X \).

A typical example of a coarse structure is the bounded coarse structure induced by a metric \( d_X : X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\} \):

\[
C_{d_X} := \{E \subseteq X \times X | \sup d_X (E) < \infty \} \cup \{\emptyset\}.
\]

Then the boundedness defined above agrees with the usual boundedness. We assume that every metric space is endowed with the bounded coarse structure throughout this paper.

Let \( f, g : X \to Y \) be maps from a set \( X \) to a coarse space \( Y \). We say that \( f \) and \( g \) are bornotopic (or close) if \( (f \times g)(\Delta_X) \in C_Y \). Obviously bornotopy gives an equivalence relation on the set \( Y^X \) of all maps from \( X \) to \( Y \).

Suppose \( f : X \to Y \) is a map between coarse spaces \( X, Y \). Then \( f \) is said to be

1. proper if \( f^{-1} (B) \in B_X \) for all \( B \in B_Y \);
2. bornologous if \( (f \times f)(E) \in C_Y \) for all \( E \in C_X \);
3. coarse if it is proper and bornologous;
4. an asymorphism (or an isomorphism of coarse spaces) if it is a bornologous bijection such that the inverse map is also bornologous;
5. a coarse equivalence (or a bornotopy equivalence) if it is bornologous, and there exists a bornologous map \( g : Y \to X \) (called a coarse inverse or a bornotopy inverse of \( f \)) such that \( g \circ f \) and \( f \circ g \) are bornotopic to the identity maps \( \text{id}_X \) and \( \text{id}_Y \), respectively.

For more information, see the monograph [7] by John Roe.

2 Coarsely connected components

Let \( X \) be a coarse space. A subset \( A \) of \( X \) is said to be coarsely connected if \( \{x, y\} \in B_X \) for all \( x, y \in A \) ([7, Definition 2.11]). For \( x \in X \), we set

\[ Q_X (x) := \bigcup_{x \in B_X} B, \]

and call it the coarsely connected component of \( X \) containing \( x \). It is easy to see that \( Q_X (x) \) is the largest coarsely connected subset of \( X \) that contains \( x \) (see also [7, Remark 2.20]). We denote the set of all coarsely connected components of \( X \) by \( Q(X) \):

\[ Q(X) := \{Q_X (x) | x \in X\}. \]

**Lemma 1.** Let \( f : X \to Y \) be a bornologous map. If \( X \) is coarsely connected, then so is the image \( f(X) \).

**Proof.** The statement is immediate from the fact that every bornologous map preserves boundedness.

**Theorem 2** (Functoriality). Every bornologous map \( f : X \to Y \) functorially induces a map \( Q(f) : Q(X) \to Q(Y) \) by \( Q(f)(Q_X (x)) := Q_Y(f(x)) \).

**Proof.** It suffices to verify the well-definedness. Let \( x, y \in X \) and suppose \( Q_X (x) = Q_X (y) \). Since \( f \) is bornologous and \( Q_X (x) \) is coarsely connected, \( f(Q_X (x)) \) is coarsely connected and contains \( f(x) \). By the maximality of \( Q_Y(f(x)) \), we have that \( f(y) \in f(Q_X (y)) = f(Q_X (x)) \subseteq Q_Y(f(x)) \). By the maximality of \( Q_Y(f(y)) \), we have that \( Q_Y(f(x)) \subseteq Q_Y(f(y)) \). By symmetry, \( Q_Y(f(y)) \subseteq Q_Y(f(x)) \) holds. Therefore \( Q_Y(f(x)) = Q_Y(f(y)) \).

**Theorem 3** (Coarse invariance). If two bornologous maps \( f, g : X \to Y \) are bornotopic, then \( Q(f) = Q(g) \).

**Proof.** The proof is similar to that of Theorem 2. Let \( x \in X \). Since \( f \) and \( g \) are bornotopic, \( \{f(x), g(x)\} \in (f \times g)(\Delta_X) \subseteq C_Y \), so \( \{f(x), g(x)\} \) is bounded in \( Y \). Thus \( \{f(x), g(x)\} \) is coarsely connected and contains \( f(x) \). By the maximality of \( Q_Y(f(x)) \), we have that \( g(x) \in \{f(x), g(x)\} \subseteq Q_Y(f(x)) \). By the maximality of \( Q_Y(g(x)) \), we have that \( Q_Y(f(x)) \subseteq Q_Y(g(x)) \). The reverse inclusion \( Q_Y(g(x)) \subseteq Q_Y(f(x)) \) holds by symmetry. It follows that \( Q(f)(Q_X (x)) = Q_Y(f(x)) = Q_Y(g(x)) = Q(g)(Q_X (x)) \).
3 Metrisation of $S(X,\xi)$

Let $(X,\xi)$ be a pointed coarse space. A coarse map $s : (\mathbb{N},0) \to (X,\xi)$ is called a coarse sequence in $(X,\xi)$. Denote by $S(X,\xi)$ the set of all coarse sequences of $(X,\xi)$. In the preceding studies [6, 4, 3, 2], $S(X,\xi)$ is just a set with no structure. In fact, as we shall see below, $S(X,\xi)$ has a geometric structure relevant to $\sigma(X,\xi)$. We define a metric $d_{S(X,\xi)} : S(X,\xi) \times S(X,\xi) \to \mathbb{N} \cup \{\infty\}$ on $S(X,\xi)$ as follows:

$$d_{S(X,\xi)}(s,t) = \inf\{ n \in \mathbb{N} | (s,t) \in \{ \equiv_{X,\xi} \cup \equiv_{X,\xi} \}^n \},$$

where $\inf \emptyset := \infty$. It is easy to check that $d_{S(X,\xi)}$ is a metric. Thus $S(X,\xi)$ is equipped with a coarse structure, viz., the bounded coarse structure induced by $d_{S(X,\xi)}$.

Lemma 4. Let $(X,\xi)$ be a pointed coarse space and $s,t \in S(X,\xi)$.

1. The following are equivalent:
   (a) $s \equiv_{X,\xi} t$;
   (b) $d_{S(X,\xi)}(s,t) \in \mathbb{N}$;
   (c) there exists a sequence $\{u_i\}_{i=0}^n$ in $S(X,\xi)$ of length $n+1$ such that $u_0 = s$, $u_n = t$ and $d_{S(X,\xi)}(u_i, u_{i+1}) = 1$ for all $i < n$, where $n$ is an arbitrary constant greater than or equal to $d_{S(X,\xi)}(s,t)$.

2. The following are equivalent:
   (a) $s \not\equiv_{X,\xi} t$;
   (b) $d_{S(X,\xi)}(s,t) = \infty$;
   (c) there is no finite sequence $\{u_i\}_{i=0}^n$ in $S(X,\xi)$ such that $u_0 = s$, $u_n = t$ and $d_{S(X,\xi)}(u_i, u_{i+1}) = 1$ for all $i < n$.

Proof. Notice that $d_{S(X,\xi)}(s,t) \leq n$ if and only if there exists a sequence $\{u_i\}_{i=0}^n$ in $S(X,\xi)$ of length $n+1$ such that $u_0 = s$, $u_n = t$, and $d_{S(X,\xi)}(u_i, u_{i+1}) = 1$ for all $i < n$. Also, note that $d_{S(X,\xi)}(s,t) = \infty$ if and only if there is no such finite sequence in $S(X,\xi)$. The above equivalences are now obvious. \qed

Theorem 5 (Functoriality). Each coarse map $f : (X,\xi) \to (Y,\eta)$ functorially induces a bornologous map $S(f) : S(X,\xi) \to S(Y,\eta)$ by $S(f)(s) := f \circ s$.

Proof. Well-definedness: let $s \in S(X,\xi)$. Clearly $S(f)(s)$ is a map from $(\mathbb{N},0)$ to $(Y,\eta)$. The class of coarse maps is closed under composition, so $S(f)(s)$ is coarse. (Let $E \in C_Y$. Then $(s \times s)(E) \in C_X$ by the bornologousness of $s$, so $(f \circ s \times f \circ s)(E) = (f \times f)((s \times s)(E)) \in C_Y$ by the bornologousness of $f$. Let $B \in B_Y$. Then $f^{-1}(B) \in B_X$ by the properness of $f$, and hence $(f \circ s)^{-1}(B) = s^{-1} \circ f^{-1}(B) \in B_X$ by the properness of $s$.) Hence $S(f)(s) \in S(Y,\eta)$.

Bornologousness: Let $s,t \in S(X,\xi)$ and suppose $d_{S(X,\xi)}(s,t) \leq n$, i.e., there is a sequence $\{u_i\}_{i=0}^n$ in $S(X,\xi)$ of length $n+1$ such that $u_0 = s$, $u_n = t$, and $d_{S(X,\xi)}(u_i, u_{i+1}) = 1$ for all $i < n$. Then the sequence $\{f \circ u_i\}_{i=0}^n$ witnesses that $d_{S(Y,\eta)}(S(f)(s), S(f)(t)) = d_{S(Y,\eta)}(f \circ s, f \circ t) \leq n$. \qed
Theorem 6 (Preservation of bornotopy). If coarse maps \( f, g : (X, \xi) \to (Y, \eta) \) are bornotopic, then so are \( S(f), S(g) : S(X, \xi) \to S(Y, \eta) \).

Proof. Let \( s \in S(X, \xi) \). We define a map \( t : (\mathbb{N}, 0) \to (Y, \eta) \) as follows:

\[
t(i) := \begin{cases} S(f)(s)(j), & i = 2j, \\ S(g)(s)(j), & i = 2j + 1. \end{cases}
\]

Let us verify that \( t \in S(Y, \eta) \). Firstly, let \( B \in \mathcal{B}_Y \). Then

\[
t^{-1}(B) = 2(S(f)(s))^{-1}(B) \cup \left(2(S(g)(s))^{-1}(B) + 1\right).
\]

Since \( S(f)(s) \) and \( S(g)(s) \) are proper, the two sets \( 2(S(f)(s))^{-1}(B) \) and \( 2(S(g)(s))^{-1}(B) + 1 \) are bounded in \( \mathbb{N} \) (i.e. finite), so \( t^{-1}(B) \in \mathcal{B}_Y \). Therefore \( t \) is proper. Secondly, let \( n \in \mathbb{N} \). Since \( S(f)(s) \) and \( S(g)(s) \) are bornologous, there exists an \( E \in \mathcal{C}_Y \) such that \( (S(f)(s)(i), S(f)(s)(j)) \in E \) and \( (S(g)(s)(i), S(g)(s)(j)) \in E \) hold for all \( i, j \in \mathbb{N} \) with \( |i - j| \leq n \). Since \( f \) and \( g \) are bornotopic,

\[
F := \{(S(f)(s)(i), S(g)(s)(i)) \mid i \in \mathbb{N}\} = \{(f \circ s(i), g \circ s(i)) \mid i \in \mathbb{N}\} \in (f \times g)(\Delta_X) \in \mathcal{C}_Y.
\]

Then \( (S(f)(s)(i), S(g)(s)(j)) \in E \circ F \in \mathcal{C}_Y \) and \( (S(g)(s)(i), S(f)(s)(j)) \in E \circ F^{-1} \in \mathcal{C}_Y \) hold for all \( i, j \in \mathbb{N} \) with \( |i - j| \leq n \). Now let \( G := E \cup (E \circ F) \cup (E \circ F^{-1}) \in \mathcal{C}_Y \). Then \( (t(i), t(j)) \in G \) holds for all \( i, j \in \mathbb{N} \) with \( |i - j| \leq n \). Therefore \( t \) is bornologous.

Both \( S(f)(s) \) and \( S(g)(s) \) are subsequences of \( t \), i.e., \( S(f)(s) \equiv t \equiv S(g)(s) \). Therefore \( S(f) \) and \( S(g) \) are bornotopic.

\( \square \)

The next theorem shows that the base point can be replaced with any other point lying in the same coarsely connected component.

Theorem 7 (Changing the base point). Let \( X \) be a coarse space, and \( \xi_1, \xi_2 \in X \). If \( Q_X(\xi_1) = Q_X(\xi_2) \), then \( S(X, \xi_1) \) and \( S(X, \xi_2) \) are isometric.

Proof. Define maps \( T_{21} : S(X, \xi_1) \to S(X, \xi_2) \) and \( T_{12} : S(X, \xi_2) \to S(X, \xi_1) \) as follows:

\[
T_{21}(s_{\xi_1}) := \begin{cases} s_{\xi_2}, & d_{S(Y, \eta)}(s_{\xi_1}, s_{\xi_2}) \leq 2, \\ 0, & \text{otherwise}. \end{cases}
\]

Let \( \overline{s_{\xi_1}} \) and \( \overline{s_{\xi_2}} \) be the base point extensions of \( s_{\xi_1} \) and \( s_{\xi_2} \), respectively. Then \( T_{21}(\overline{s_{\xi_1}}) = \overline{s_{\xi_2}} \) and \( T_{12}(\overline{s_{\xi_2}}) = \overline{s_{\xi_1}} \) are isometries.

\( \square \)
by

\[
T_{21}(s) (i) = \begin{cases} 
\xi_2, & i = 0, \\
T(s), & i > 0, 
\end{cases} 
\]

\[
T_{12}(t) (i) = \begin{cases} 
\xi_1, & i = 0, \\
T(t), & i > 0. 
\end{cases} 
\]

We first verify the well-definedness, i.e. \( T_{21}(s) \in S(X, \xi_2) \) and \( T_{12}(t) \in S(X, \xi_1) \). Obviously \( T_{21}(s)(0) = \xi_2 \). Let \( B \in \mathcal{B}_X \). Then \( (T_{21}(s))^{-1}(B) \subseteq s^{-1}(B) \cup \{0\} \), where \( s^{-1}(B) \) is bounded in \( \mathbb{N} \) (i.e. finite) by the properness of \( s \), so \( (T_{21}(s))^{-1}(B) \) is also bounded in \( \mathbb{N} \). Hence \( T_{21}(s) \) is proper. Next, let \( E \in \mathcal{C}_X \). For each \( (i, j) \in E \), there are the following possibilities:

**Case 1.** \( i = j = 0 \).

Then \( T_{21}(s)(i) = \xi_2 = T_{21}(s)(j) \), so \( (T_{21}(s)(i), T_{21}(s)(j)) \in \Delta_X \in \mathcal{C}_X \).

**Case 2.** \( i = 0 \) and \( j \neq 0 \).

In this case, \( T_{21}(s)(i) = \xi_2, s(i) = \xi_1 \) and \( s(j) = T_{21}(s)(j) \).

So \( (T_{21}(s)(i), T_{21}(s)(j)) \in \{ (\xi_2, \xi_1) \} \circ (s \times s)(E) \subseteq \mathcal{C}_X \).

**Case 3.** \( i \neq 0 \) and \( j = 0 \).

Similar to the above case, we have \( (T_{21}(s)(i), T_{21}(s)(j)) \in (s \times s)(E) \circ \{ (\xi_1, \xi_2) \} \subseteq \mathcal{C}_X \).

**Case 4.** \( i \neq 0 \) and \( j \neq 0 \).

Then \( T_{21}(s)(i) = s(i) \) and \( T_{21}(s)(j) = s(j) \), whence we have \( (T_{21}(s)(i), T_{21}(s)(j)) \in (s \times s)(E) \subseteq \mathcal{C}_X \).

Set \( F := \Delta_X \cup \{ (\xi_2, \xi_1) \} \circ (s \times s)(E) \cup \{ (\xi_1, \xi_2) \} \subseteq \mathcal{C}_X \).

Then \( (T_{21}(s), T_{21}(s))(E) \subseteq F \in \mathcal{C}_X \), so \( (T_{21}(s), T_{21}(s))(E) \subseteq \mathcal{C}_X \). Hence \( T_{21}(s) \) is bornologous. Since the definitions are symmetric, the same argument applies to \( T_{12}(t) \).

Clearly \( T_{12} \circ T_{21} = \text{id}_{S(X, \xi_1)} \) and \( T_{21} \circ T_{12} = \text{id}_{S(X, \xi_2)} \). It suffices to prove that \( T_{21} \) is an isometry.

Let \( s, t \in S(X, \xi_1) \) and suppose that \( s \equiv X, \xi t \), i.e., there is a strictly monotone function \( \kappa : \mathbb{N} \to \mathbb{N} \) such that \( s = t \circ \kappa \). Since \( \kappa(i) \geq i \), we have \( T_{21}(s)(i) = s(i) = t(\kappa(i)) = T_{21}(t)(\kappa(i)) \) for all \( i > 0 \). Now, define

\[
\kappa'(i) = \begin{cases} 
0, & i = 0, \\
\kappa(i), & i > 0. 
\end{cases} 
\]

Then \( T_{21}(s)(i) = T_{21}(t)(\kappa'(i)) \) holds for all \( i \in \mathbb{N} \) (including the case \( i = 0 \)). Hence \( T_{21}(s) \equiv X, \xi T_{21}(t) \). Note that, by symmetry, the same applies to \( T_{12} \). Conversely, let \( s, t \in S(X, \xi_1) \) and suppose \( T_{21}(s) \equiv X, \xi T_{21}(t) \). Then \( s = T_{12} \circ T_{21}(s) \equiv X, \xi T_{12} \circ T_{21}(t) = t \).

Now, let \( s, t \in S(X, \xi_1) \) and suppose \( d_{S(X, \xi_1)}(s, t) \leq n \), i.e., there is a
sequence \( \{u_i\}_{i=0}^n \) in \( S(X,\xi) \) of length \( n+1 \) such that \( u_0 = s, u_n = t, \) and \( u_i \in X,\xi \) or \( u_i+1 \in X,\xi \) for all \( i < n \). By the previous paragraph, \( T_{21}(u_i) \in X,\xi \) or \( T_{21}(u_{i-1}) \in X,\xi \) for all \( i < n \). So \( d_{S(X,\xi)}(T_{21}(s),T_{21}(t)) \leq n \). The same applies to \( T_{12} \) by symmetry. Conversely, let \( s,t \in S(X,\xi) \) and suppose \( d_{S(X,\xi)}(T_{21}(s),T_{21}(t)) \leq n \). Then it follows that \( d_{S(X,\xi)}(s,t) = d_{S(X,\xi)}(T_{21} \circ T_{21}(s),T_{12} \circ T_{21}(t)) \leq n \). Consequently, both \( T_{21} \) and \( T_{12} \) are isometries.

In fact, the metric function \( d_{S(X,\xi)} \) only takes the values 0, 1, 2 and \( \infty \). To show this fact, we need the “confluence” property of \( \pmb{21} \).

**Lemma 8** (DeLyser, LaBuz and Tobash [2, Lemma 3.1]). Let \( s,t,u \in S(X,\xi) \) and suppose \( s \in X,\xi \) \( t,u \). Then there is a \( v \in S(X,\xi) \) such that \( t,u \in X,\xi \) \( v \).

![Diagram](image)

**Proof.** By the definition of “subsequence”, there are strictly monotone functions \( \kappa,\lambda : \mathbb{N} \rightarrow \mathbb{N} \) such that \( s = t \circ \kappa = u \circ \lambda \). The desired sequence \( v \in S(X,\xi) \) is given by

\[
\begin{align*}
&\overbrace{s(0),t(1),\ldots,t(\kappa(1)-1),s(1)}^{t(0),\ldots,t(\kappa(1))}, \overbrace{s(0),u(1),\ldots,u(\lambda(1)-1),s(1)}^{u(0),\ldots,u(\lambda(1))}, \\
&\overbrace{s(1),t(\kappa(1)+1),\ldots,t(\kappa(2)-1),s(2)}^{t(\kappa(1)),\ldots,t(\kappa(2))}, \overbrace{s(1),u(\lambda(1)+1),\ldots,u(\lambda(2)-1),s(2)}^{u(\lambda(1)),\ldots,u(\lambda(2))}, \\
&\vdots
\end{align*}
\]

Obviously \( v \) has \( t \) and \( u \) as subsequences. Let \( E = \{(i,j)\mid |i-j| \leq 1\} \).
(Note that \( \mathbb{C}_E \) is generated by \( \{E^n\mid n \in \mathbb{N}\} \).) Since \( s, t \) and \( u \) are bornologous, \( (s \times s)(E), (t \times t)(E), (u \times u)(E) \in \mathbb{C}_E \). Note that any two adjacent points \( (v(i),v(i \pm 1)) \) are one of the following forms:

\[
(t(j),t(j \pm 1)), (s(j),s(j \pm 1)), (u(j),u(j \pm 1)), (s(j),s(j)),
\]

so \( (v \times v)(E) \subseteq (t \times t)(E) \cup (s \times s)(E) \cup (u \times u)(E) \cup \Delta_X \in \mathbb{C}_E \). Hence \( v \) is bornologous. Similarly, one can easily prove that \( v \) is proper (i.e. diverges to infinity).\(\square\)
Lemma 9 (DeLyser, LaBuz and Tobash [2, Proposition 3.2]). Let \( s, t \in S(X, \xi) \) and suppose \( s \equiv X,\xi t \). Then there is a \( u \in S(X, \xi) \) such that \( s, t \preceq X,\xi u \).

Proof. Choose a sequence \( \{u_i\}_{i=0}^n \) in \( S(X, \xi) \) such that \( u_0 = s, u_n = t, \) and \( u_i \not\equiv X,\xi u_{i+1} \) or \( u_{i+1} \not\equiv X,\xi u_i \) for all \( i < n \). We show that there is a \( v \in S(X, \xi) \) such that \( u_0, u_n \preceq X,\xi v \) by induction on the length \( n \). The base case \( n = 0 \) is trivial. Suppose \( n > 0 \). Since \( u_0 \equiv X,\xi u_{n-1} \), there is a \( v \in S(X, \xi) \) such that \( u_0, u_{n-1} \preceq X,\xi v \) by the induction hypothesis.

Case 1. \( u_{n-1} \not\equiv X,\xi u_n \).
Since \( u_{n-1} \not\equiv X,\xi u_n, v \), there is a \( v' \in S(X, \xi) \) such that \( u_n, v \equiv X,\xi v' \) by Lemma 8. Then \( u_0 \equiv X,\xi v \equiv X,\xi v' \), so \( u_0 \equiv X,\xi v' \).

Case 2. \( u_{n-1} \equiv X,\xi u_n \).
Then \( u_n \equiv X,\xi u_{n-1} \equiv X,\xi v \), so \( u_n \equiv X,\xi v \).

Theorem 10. \( d_S(X,\xi) : X \times X \rightarrow \{0, 1, 2, \infty\} \).

Proof. Let \( s, t \in S(X, \xi) \) and suppose \( s \equiv X,\xi t \). There is a \( u \in S(X, \xi) \) such that \( s, t \preceq X,\xi u \) by Lemma 9. Hence \( d_S(X,\xi)(s, t) \leq 2 \).

A similar argument in Lemma 9 is often used in the context of rewriting systems (such as lambda calculus). See also [1, Chapter 6].
4 Alternative definition of $\sigma$

Our main theorem is the following. This gives an alternative definition of $\sigma(X, \xi)$ in terms of the coarse structure of $S(X, \xi)$.

**Theorem 11.** Let $(X, \xi)$ be a pointed coarse space. Then $[s]_{X, \xi}^\sigma = Q_{S(X, \xi)}(s)$ for all $s \in S(X, \xi)$. Hence $\sigma(X, \xi) = Q(S(X, \xi))$.

*Proof.* Let $s \in S(X, \xi)$. Then, by Lemma 4-(1), $[s]_{X, \xi}^\sigma$ is coarsely connected (in fact, 1-chain-connected) as a subset of $S(X, \xi)$, and contains $s$. Hence $[s]_{X, \xi}^\sigma \subseteq Q_{S(X, \xi)}(s)$ by the maximality of $Q_{S(X, \xi)}(s)$. Conversely, let $t \in Q_{S(X, \xi)}(s)$. By Lemma 4-(2), $s \equiv_{X, \xi} t$ must hold, and therefore $t \in [s]_{X, \xi}^\sigma$. Hence $Q_{S(X, \xi)}(s) \subseteq [s]_{X, \xi}^\sigma$. \hfill $\Box$

This theorem yields quite simple and systematic proofs of some existing results on $\sigma(X, \xi)$.

**Theorem 12.** Each coarse map $f : (X, \xi) \to (Y, \eta)$ functorially induces a map $\sigma(f) : \sigma(X, \xi) \to \sigma(Y, \eta)$ by $\sigma(f)([s]_{X, \xi}^\sigma) := [f \circ s]_{Y, \eta}^\sigma$.

*Proof.* Immediate from Theorem 2, Theorem 5 and Theorem 11. \hfill $\Box$

**Corollary 13** (Miller, Stibich and Moore [6, Theorem 10]). If pointed coarse spaces $(X, \xi)$ and $(Y, \eta)$ are asymorphic, then $\sigma(X, \xi) \cong \sigma(Y, \eta)$.

*Proof.* Obvious from the fact that every functor preserves isomorphisms. \hfill $\Box$

**Theorem 14.** If coarse maps $f, g : (X, \xi) \to (Y, \eta)$ are bornotopic, then $\sigma(f) = \sigma(g)$.

*Proof.* Immediate from Theorem 6, Theorem 3 and Theorem 11. \hfill $\Box$

**Corollary 15** (DeLyser, LaBuz and Wetsell [3, Theorem 4]). If pointed coarse spaces $(X, \xi)$ and $(Y, \eta)$ are coarsely equivalent, then $\sigma(X, \xi) \cong \sigma(Y, \eta)$.

*Proof.* Let $f : (X, \xi) \to (Y, \eta)$ be a coarse equivalence with a coarse inverse $g : (Y, \eta) \to (X, \xi)$. Then $f \circ g$ and $g \circ f$ are bornotopic to $id_{(Y, \eta)}$ and $id_{(X, \xi)}$, respectively. By Theorem 12 and Theorem 14,

$$id_{\sigma(Y, \eta)} = \sigma(id_{(Y, \eta)})$$

$$= \sigma(f \circ g)$$

$$= \sigma(f) \circ \sigma(g),$$

$$id_{\sigma(X, \xi)} = \sigma(id_{(X, \xi)})$$

$$= \sigma(g \circ f)$$

$$= \sigma(g) \circ \sigma(f),$$

so $\sigma(f)$ and $\sigma(g)$ are inverse to each other. Hence $\sigma(X, \xi) \cong \sigma(Y, \eta)$. \hfill $\Box$

**Corollary 16** (DeLyser, LaBuz and Wetsell [3, Proposition 3]). Let $X$ be a coarse space, and $\xi_1, \xi_2 \in X$. If $Q_X(\xi_1) = Q_X(\xi_2)$, then $\sigma(X, \xi_1)$ and $\sigma(X, \xi_2)$ are equipotent (i.e. have the same cardinality).
Proof. By Theorem 7, $S(X, \xi_1)$ and $S(X, \xi_2)$ are isometric and hence isomorphic. So $\sigma(X, \xi_1) = Q(S(X, \xi_1)) \cong Q(S(X, \xi_2)) = \sigma(X, \xi_2)$ by Theorem 2 and Theorem 11.

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Bibliography


