An energy balance model of carbon's effect on climate change

Lucas Benney and Anca Rădulescu

Lucas Benney graduated in 2015 with a degree in Physics from SUNY New Paltz. He is currently working at Draper Laboratory in Cambridge Massachusetts.

Anca Rădulescu became an Assistant Professor in Mathematics at SUNY New Paltz in Fall 2014. Her research interests are in dynamics and in modeling systems from the natural sciences.

Abstract Due to climate change, the interest in studying our climatic system using mathematical modeling has become tremendous in recent years. One well-known model is Budyko's system, which represents the coupled evolution of two variables, the ice-line and the average Earth surface temperature. The system depends on natural parameters, such as the Earth albedo, and the amount A of carbon in the atmosphere.

We introduce a 3-dimensional extension of this model in which we regard A as the third coupled variable of the system. We analyze the phase space and dependence on parameters, looking for Hopf bifurcations and the birth of cycling behavior. We interpret the cycles as climatic oscillations from a hot Earth with high iceline to a cold Earth with low iceline. We illustrate how long-term oscillations between an ice-free and an ice-ball Earth can be crucially perturbed by even small changes in human CO_2 emission patterns, with predictions aligned with current theories of global warming.

Introduction

Earth's climate is constantly changing. This change has become much more prominent over the past century, during which time the average surface temperature of the Earth has increased by almost a full degree Celsius. Figure 1 shows together the average annual surface temperature of the Earth (on the top), and the annual concentration of carbon dioxide (on the bottom). Both graphs show a similar, almost exponential

increase with time. The correlation between the two trends suggests that surface temperature is related to carbon dioxide levels.
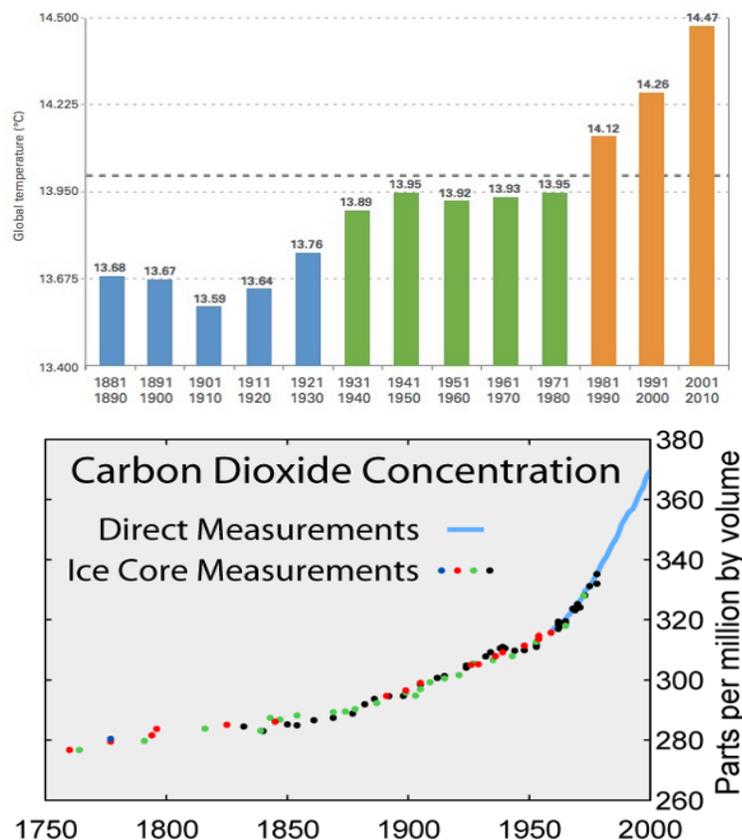
Figure 1: **Earth's average temperature and carbon dioxide concentration annually. Top.** *Earth's average surface temperature over the past century.* **Bottom.** *Measured levels of carbon dioxide concentrations annually.*

Over the past hundreds of millions of years the Earth's average surface temperature has gone through many different cycles. There have been times where it was so warm that reptiles could survive above the arctic circle and times when glaciers covered most of the Earth. The average surface temperature has been rising consistently over the past hundred years. One important question is whether this increase is a steady trend, or whether it is the upper side of a cyclic pattern. In either case, how long until this high maximum temperature value makes Earth uninhabitable? Can our behavior, such as green house gas emissions or excess use of fossil fuels, change the established cycles?

The Budyko model

Russian Climatologist Mikhail Budyko [1] formulated the following integro-differential equation (1) by assuming Earth's annual average surface temperature at a given latitude $T = T(t, y)$ (in ^oC) can be represented as a function of time (t) and the sine of the latitude ($y = \sin(\theta)$):

$$R\frac{\partial T}{\partial t} = Qs(y)(1 - \alpha(y, \eta)) - (A + BT) + C(\overline{T} - T) \tag{1}$$

where

$$\overline{T}(t) = \int_0^1 T(t, y) \, dy.$$

This energy balance model (EBM), referred to as Budyko's model [10], is based on simplifications of factors that cause the Earth's surface temperature to change. The temperature is assumed to be constant on a given latitude circle and symmetric across the equator (hence y ranges from 0 at the equator to 1 at the north pole). It is assumed that the average temperature is a continuous function with respect to time. $\overline{T}$ is the global annual mean temperature.

Both sides of the equation are expressed in W/m^2 (since the parameters R, representing the average heat capacity of the Earth's surface, is measured in $J/m^2 \, ^\circ C$, and the temperature T is measured in $^\circ C$. Parameter values and their corresponding units are shown in Table 1.

A key component of the equation is $\alpha(y, \eta)$, which represents the albedo (amount of shortwave solar radiation reflected back into space) for the Earth, which is defined by

$$\alpha(y, \eta) = \left\{ \begin{array}{ll} \alpha_w, & \text{if } y < \eta, \\ \alpha_s, & \text{if } y > \eta. \end{array} \right. \tag{2}$$

The albedo function is defined piecewise and depends on the latitude y and on the position η of the iceline (the glacier line), so that for $y > \eta$ the Earth is covered in ice, with an albedo of $\alpha_s = 0.62$, and for $y < \eta$ the Earth is ice-free, with $\alpha_w = 0.32$. More radiation is reflected for ice than water, producing a larger albedo α for above-iceline latitudes.

The parameter Q represents the "solar constant," or the annual global mean insolation. The distribution of that insolation over latitude is represented by $s(y)$, with $\int_0^1 s(y) \, dy = 1$. Putting these pieces together, one can interpret the first term on the right of (1) as the amount of incoming solar energy absorbed by the Earth.

Outgoing longwave radiation (OLR) is approximated by the second term on the right $A + BT$ (the values of A and B, determined by satellite measurements, are shown in Table 1). This term can be looked at as a measure of the amount of carbon (or more generally, greenhouse gases) in the atmosphere. If there is more carbon in the atmosphere, less radiation escapes, therefore this term will be smaller.

The final term on the right represents convection, i.e., heat transfer between latitudes. In Budyko's model it is assumed that over the period of a year this transfer of heat can be represented as the difference between the average global surface temperature $\overline{T}$ and the surface temperature at the current latitude, then multiplying this difference by a proportionality constant (C) (also determined by satellites).

Iceline dynamics

More recent work by McGehee and Widiasih incorporated temperature-triggered iceline variability in the original EBM, and formulated the temperature-iceline dynamics

Parameter	Value	Units
Q	343	W/m^2
s_2	-0.482	Dimensionless
A	202	W/m^2
B	1.9	W/m^2 $^\circ C$
D	3.04	W/m^2 $^\circ C$
R	4×10^9	J/m^2 $^\circ C$
α_w	0.32	Dimensionless
α_s	0.62	Dimensionless
T_c	-10	$^\circ C$
Ω	$1.5 \cdot 10^{11}$	J/m^2
ε	$3.9 \cdot 10^{-13}$	

Table 1: **Model parameter values and units**, *as per the original references.*

as a system of two coupled differential equations [9, 10]. Below, we only briefly outline the derivation of the coupled equations; a comprehensive explanation can be found in the original reference.

First, a simpler approximation was introduced for the distribution of insolation $s(y)$, which was expressed in terms of the angle β between the Earth's axis of rotation and perpendicular to plane of Earth's orbit [8]. A quadratic approximation (to within 2% error) led to an expression for $s(y)$ of the form:

$$s(y) \approx 1 + \frac{s_2}{2}\left(3y^2 - 1\right) \tag{3}$$

It was shown [8] that $s_2 \approx -0.482$. This approximation allowed for simpler methods of solving for equilibrium solutions $T_\eta^*(y)$ to Budyko's model for a fixed iceline η, without taking a large toll on accuracy. The discontinuity of the albedo function $\alpha(y, \eta)$ at $y = \eta$ leads to the question of defining $T_\eta^*(y)$ at $y = \eta$. This was taken to be the average of the two side limits:

$$T_\eta^*(\eta) = \frac{T_\eta^*(\eta-) + T_\eta^*(\eta+)}{2}$$

leading to the equilibria $T_\eta^*(y)$.

It has been observed that glaciers form at a temperature of approximately $-10^\circ C$. So if it is assumed that the ice line is stationary, the average temperature across that iceline should be a critical temperature, $T_c = -10^\circ C$. However, in light of observations of iceline dynamics over many years, the stationarity assumption is not realistic [8].

That is because if the equilibrium temperature profile is greater than the critical temperature, the ice line will retreat towards $\eta = 1$ and the opposite if the temperature is below T_c. According to these considerations, Widiasih [16] introduced the rate of change of the ice line latitude as being proportional to the difference between the equilibrium temperature profile and the critical temperature (with proportionality constant ε extremely small, since it represents the time scale of glacier dynamics):

$$\frac{d\eta}{dt} = \varepsilon(T_\eta^*(\eta) - T_c) \tag{4}$$

The energy required to melt ice was also included in the original EBM, via a

parameter Ω, representing the amount of energy required to melt a square meter of ice:

$$\Omega = 1.5 \cdot 10^{11} \, J/m^2 \tag{5}$$

With these extensions, the Budyko model with iceline dynamics becomes a system of coupled equations in (T, η):

$$
\begin{aligned}
\frac{d\eta}{dt} &= \varepsilon(T_\eta^*(\eta) - T_c) \\
R\frac{\partial T}{\partial t} &= Qs(y)(1 - \alpha(y,\eta)) - (A + BT) + C(\overline{T} - T) - \Omega\frac{d\eta}{dt}
\end{aligned}
\tag{6}
$$

In the original reference [10], the authors use a Legendre expansion and change of variables to find a two dimensional invariant subspace that yields to a simple system of two coupled differential equations:

$$
\begin{aligned}
\frac{d\eta}{dt} &= \varepsilon\left[w + \frac{Qs_2}{2}\left(1 - \frac{\alpha_w + \alpha_s}{2}\right)\left(\frac{3\eta^2 - 1}{B + C}\right) - T_c\right] \\
R\frac{dw}{dt} &= Q\left(1 - \frac{\alpha_w + \alpha_s}{2}\right) - A \\
&\quad + \frac{QC}{B + C}(\alpha_s - \alpha_w)\left(\eta - \frac{1}{2} + \frac{s_2}{2}(\eta^3 - \eta)\right) - Bw - \Omega\frac{d\eta}{dt}.
\end{aligned}
\tag{7}
$$

Nonlinear dynamics

These two coupled equations define a two-dimensional, continuous time, nonlinear dynamical system. A typical analysis of such a system begins with finding nullclines and equilibria, with a linear analysis around equilibria to establish local stability, and may continue with a search for global stability features, with searching for limit cycles and with an analysis of the system's dependence on parameters. Often times, however, even seemingly simple nonlinear systems may be too complicated to solve directly, and this plan of action may fail, due to the difficulty of solving nonlinear algebraic equations, right at the first step: that of locating equilibria. In this case, numerical methods may offer adequate support (although numerical algorithms themselves may be problematic in the context of certain nonlinear features).

The two-dimensional system (7) was approached with a combination of analytical and numerical computations to find the system's two equilibria, estimate their position and determine their stability (one was found to be a node with $\eta \sim 0.95$, and one a saddle with $\eta = 0.25$). The authors also study the sensitivity of the system with respect to the parameter A (see Figure 2), to find a saddle node *bifurcation*. A bifurcation is a state of the system that represents a sharp transition from one dynamic behavior to another, which may occur with changes in the number of equilibria and their stability, or with the formation or dissolution of a limit cycle (in the **Numerical simulations** section, we will show more examples of different types of bifurcations).

The bifurcation diagram of the system (7) with respect to A (shown in Figure 2) describes a sharp transition at a tipping point of $A \sim 212$. For values of A smaller than $A \sim 198$, the system evolves away from the only unstable equilibrium in the natural range, and converges to an iceline $\eta = 1$ (ice free Earth). For values of the parameter larger than $A \sim 212$, the system has no equilibria, and the system is driven to $\eta = 0$ (snowball Earth). For intermediate values of A, the system will settle to a high iceline, and a livable average surface temperature of about 5 $°C$. This
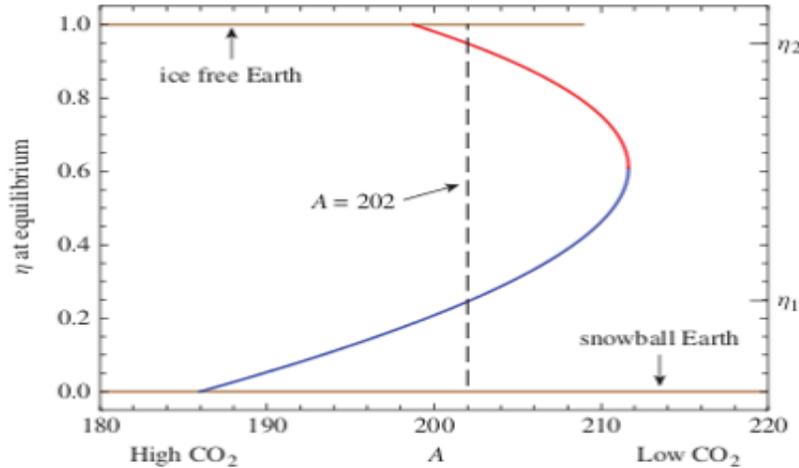
Figure 2: **Saddle node bifurcation with respect to** A. *The red plot represents a stable equilibrium curve w.r.t. A, and the blue curve an unstable (saddle) equilibrium. They collide at the bifurcation point $A = 212$ [15].*

implies, as expected, that excessive CO_2, reflected in low values of A, leads to an excessively hot Earth, but also suggests that too low values of CO_2, reflected in high A, lead to an equally undesirable outcome (frozen Earth). The existence of a range of CO_2 (and subsequently A) producing optimal temperatures for sustaining life on Earth motivated our interest in studying the coupling between CO_2 levels and surface temperature, and the effects of this coupling on the system dynamics.

We worked on conceptually extending the original results in [9] by introducing A as a third variable in the system, then we investigated the nonlinear behavior of this extension. We focused in particular on the system's dependence on parameters, and on establishing whether it has any codimension one bifurcations (i.e., sharp transitions in dynamics that appear when only varying one parameter of the system). We looked in particular for saddle-node and Hopf-fold bifurcations (see [7] for comprehensive definitions):

A *saddle node* (or *limit point*) bifurcation is a local bifurcation where two equilibria with different stability (typically a saddle and a node) collide and disappear (see Figure 2).

A *Hopf bifurcation* is a local bifurcation where an equilibrium point changes stability, with the birth of a limit cycle. This happens if an eigenvalue λ of the system's Jacobian matrix around the fixed point traverses the imaginary axis, i.e., changes from having $Re(\lambda) > 0$ to a $Re(\lambda) < 0$, with the bifurcation occurring at $Re(\lambda) = 0$. A Hopf bifurcation can be supercritical (with a stable spiral equilibrium changing stability with the birth of a stable cycle) or subcritical (with an unstable equilibrium changing stability with the birth of an unstable cycle). One can establish the type of a Hopf bifurcation by considering the quadratic nonlinear terms of the system and compute the Lyapunov coefficient σ, which is $\sigma < 0$ at a supercritical Hopf, and $\sigma > 0$ at a subcritical Hopf. When $\sigma = 0$, the bifurcation is *critical*, and the fixed point changes

stability with formation of infinitely many cycles (as previously described in [5] and [6]).

Since the direct computation of all the dynamic invariants that would permit classifications of these bifurcations would be practically intractable, we used the Matcont software package to assist our analysis (see the **Numerical simulations** section). Matcont [4] uses numerical continuation algorithms to track the changes in the behavior of the system as parameters are changed. In addition, we used direct computation of phase space trajectories for specific parameter values, in order to verify the presence of more subtle phenomena (such as critical Hopf points, which are notoriously difficult to locate by automated bifurcation-searching packages [6]).

Introducing dependence of carbon on temperature

To begin constructing our extension of the classical model (7), we start with the assumption that accumulation of greenhouse gases has a strong effect on the Earth OLR. Indeed, CO_2 absorbs energies with a wavelength of around 15 micrometers very easily. This happens to be included in the infrared region of the spectrum of light which has wavelengths ranging from 700 nanometers - 1 millimeter. This range of wavelengths is within the OLR wavelengths assumed in Budkyo's model (see [3]). CO_2 is circulated throughout our atmosphere in many ways. The main four processes are: decomposition of organic materials, respiration, photosynthesis, and combustion. Other than planting trees (or breathing less) the only control we humans have on the levels of CO_2 in the atmosphere is combustion (which we are clearly overdoing as a species). Although CO_2 is only present at a level of approximately 400 ppm (parts per million), small changes in CO_2 levels can have enormous effect on the Earth's climate.

Recall that in our model the term $A + BT$, representing outgoing longwave radiation (OLR), depends on the level of greenhouse gases, which clearly affect OLR. Observing that CO_2 is by far the dominant greenhouse gas, we regard A as a measure of the levels of CO_2 in the atmosphere, with lower values of A representing high atmospheric levels of CO_2, and high values of A making low CO_2. Since the levels of CO_2 are changing in time (as suggested by Figure 1), A will also be time dependent; we introduce $A(t)$ as a new (third) coupled variable in our dynamic model. Supported by the correlation between the graphs in the two figure panels, we suggest that A depends implicitly on time by depending directly on the surface temperature. Indeed, a variety of studies have been quantifying with increasing success over the years the dependence of the carbon footprint on temperature [14].

For example, there are geography-specific seasonal oscillations in levels of CO_2, believed to be based both on human activity and the life cycles of vegetation [13]. In the geographic zones where winter exists, photosynthesis slows during the winter months, causing less CO_2 to be removed from the atmosphere, followed by a fast increase in spring and summer, when the vegetation recovers and restarts photosynthesizing. In fall, large portions of the vegetation die and, through the decay process, emit large amounts of CO_2 back into the atmosphere. Humans, in turn, affect CO_2 levels by emitting large amounts of greenhouse gases (through maintenance needs like heating or cooling, transportation, industrial and agricultural activities). These also fluctuate throughout the year.

For our model, however, we are not primarily interested in studying annual patterns in the the CO_2 levels in specific geographic zones, but rather more global and longer-term effects produced by the coupling between temperature ($w(t)$, in our

model) and CO_2 ($A(t)$, in our model). While a precise quantitative dependence of A on w has not yet been established, and would require considerable effort to express accurately, the literature in the field provides strong evidence of specific qualitative trends. For example, it was noted that the carbon footprint differs by country, which is likely consequential primarily to climatic factors and resource availability. Recent climatology studies have suggested accounting for geography and climate in ranking countries' CO_2 emissions [12, 11]. Cooling in the hot summer months raises significantly the consumption of electricity, which is the primary cause of raising CO_2, as does heating during cold winters (some believe that the excessive cold during recent winters is responsible for some of the 2013/4 raise in CO_2 emissions reported by the US Energy Infomation Administration). The relationship between climate and carbon emissions is clearly mediated by the human response and resource availability, but other factors have been discussed, such as the influences of climate on the Earth's biota, which in turn contributes to driving CO_2 levels [12].

Incorporating these ideas in our model, we specifically assume the following: Within an ideal range of temperatures (centered around $w_0 = 20^\circ C$), live organisms require less energy for artificial maintenance, and rather contribute themselves to maintaining function of the whole system in an optimal range. Plants strive and efficiently remove atmospheric CO_2 through photosynthesis; people use reduced energy for heating and cooling, altogether increasing A to a reasonable range, by keeping dA/dt positive. Deviations from the optimal range for w lower the values of A: both excessively high and very low temperatures will diminish plant function and subsequently CO_2 removal, and will increase human carbon footprint, with an overall negative differential between these two effects that leads to high CO_2 levels, hence switching dA/dt to negative values and decreasing A. These effects exacerbate with larger $|w - w_0|$, up to the point where life becomes unsustainable. For a start, we will assume that, past this threshold, plants slowly die out, and eventually human activity will also extinguish, so that, with no positive or negative contributions to the CO_2 levels, dA/dt will return asymptotically to zero.

While the quantitative implementation is speculative, below we give a simplified possible shape for dA/dt that incorporates these details, with a positive peak at w_0, decreasing to two negative dips, then slowly returning asymptotically to zero. For now, we consider, for simplicity, that the deviations towards the high and low temperature range have symmetric effects on A:

$$\frac{dA}{dt} = a\left[1 - g(w - w_0)^2\right]e^{-f(w-w_0)^2} \tag{8}$$

where $w_0 = 20$, $a = 200$ and f and g are sensitivity parameters which tune the width and steepness of the graph. Figure 3 shows the graph of dA/dt plotted with respect to temperature, w, based on different values for the parameters, f and g. We will first study the behavior of the system with this particular form for temperature effects on CO_2 emissions. Based on these results (illustrated in Figures 4 and 5), as well as on a more in depth understanding of the physical underpinnings of this dependence, we then make some necessary adjustments to this analytic equation.

While the shape of the graph remains qualitatively the same, the function is highly sensitive quantitatively to small perturbations in either parameter. Since these parameters influence the dependence of carbon emissions on temperature, they encompass the effects of human behavior on this dependence. When interpreting our model, we view fine tuning of these parameters as corresponding adjustments in the human response to temperature changes. For a fixed f, changing g affects primarily the negative critical values; that is, smaller values of g lead to the graph reaching much shallower negative dips, without changing the position of the relaxation time. We
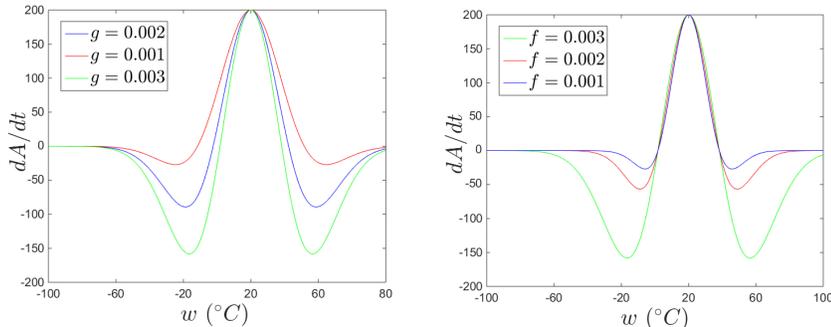
Figure 3: **Dependence of** dA/dt **on** w. **Left.** *For $f = 0.001$ and different values for g. **B.** For fixed $g = 0.003$, and different values of f.*

interpret a decrease in g as an increase in the rate of change of A around the critical temperatures corresponding to the negative dips. For a fixed g, changing f affects not only the negative critical values, but also the positions of the critical points, and the relaxation time, with imperceptible changes, however, on the graph in the positive value range. We interpret these changes as different positions for extrema of dA/dt.

Numerical simulations

The first numerical difficulty we had to overcome was inherited from the original two-dimensional fast-slow system. The huge difference of almost four orders of magnitude between the time constants R and $1/\varepsilon$ is unpleasant for both numerical and visualization purposes. Hence, before using any integration methods in our analysis, we performed a change of scale (i.e., a change in units of our time constants). Using a millennium (10^3 years $\sim 3.16 \times 10^{10}$ seconds) as our time unit, the original values of our time constants $R = 4 \cdot 10^9$, $\Omega = 1.5 \cdot 10^{11}$, $\varepsilon = 3.9 \cdot 10^{-13}$ become $R = 0.1266$, $\Omega = 0.474$, $\varepsilon = 0.01264$. All simulations are performed with these transformed values, hence the results will be expressed on a time scale measured in thousands of years.

To simulate the asymptotic behavior of our system, we used Matcont continuation algorithms for finding equilibria, extending equilibrium curves with respect to the sensitivity parameters f and g and detecting bifurcations of codimension 1 along these curves. To better understand and cautiously back-up the automated algorithms provided by Matcont (where an unoptimal choice for the integration step in conjunction with a sensitive state of the system may sometimes lead to spurious results), we used phase space trajectories for specific parameter values.

Our simulations detected rich behavior along equilibrium curves, among which saddle node and subcritical Hopf bifurcations, implying the presence of limit cycles (see Figures 4 and 5). But at a closer inspection, we found some aspects of the simulations need additional justification and tuning in order to be representative of plausible behavior. For example: both our parameters g and f tune the sensitivity of A to temperature – for a fixed f, increasing g exacerbates the negative effects of w on A, and so does increasing f when keeping g fixed.

However, unlike one may expect at a first glance, the simulated effects of increasing the parameters f and g were very different. Increasing g for fixed f pushed the system
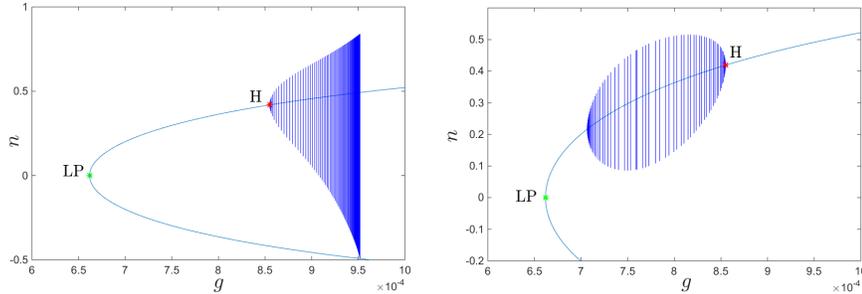
Figure 4: **Subcritical Hopf bifurcation with respect to the parameter g. Left.** *The plot shows the equilibrium curve with respect to g, for fixed $f = 0.001$. The Hopf point (which occurs approximately at $g = 0.000854$ is marked on the curve by a red star; we also illustrate the evolution of the unstable cycle sprouting from the Hopf point, as we allow g and the period to change.* **Right.** *The same equilibrium curve and Hopf point are shown; the unstable cycle is extended instead with respect to both g and f simultaneoulsy. The wider coordinate window also includes in this case a limit point occurring approximately at $g = 0.00066$ on the equilibrium curve (shown as a green star). All other system parameters were fixed for this simulation to: $B = 1.9$, $C = 3.04$, $\alpha_w = 0.32$, $\alpha_s = 0.62$, $s_2 = -0.482$, $w_0 = 20$, $T_c = -10$, $R = 0.1266$, $\Omega = 0.474$, $\varepsilon = 0.01264$, $a = 20$.*

through a Hopf bifurcation and changed the stability of the equilibrium from repelling to attracting spiral coexisting with an unstable cycle (Figure 4). This was surprising, since one rather expects that, by increasing the sensitivity to CO_2, the system would become less stable. On the other hand, the effect was the opposite when decreasing f for fixed g, and the equilibrium undergoes a Hopf bifurcation and becomes stable (Figure 5). These differences may be due to the fact that g affects the dA/dt curve along the whole effective range of temperatures, and f is more focused at the center of the domain, hence they may affect differently the dynamics within our phase and parameter ranges.

Altogether, we realized that our construction needed more clarification, and a more qualitatively accurate set of sensitivity parameters to describe the dependence of dA/dt on w. In this next section, we present a refinement of our model.

Introducing water vapor

One major effect which was omitted in our first extension was the difference between physical and chemical behavior of elements at high and low temperature extremes. Unlike low temperatures, high surface temperature raise the amounts of vapors present in the atmosphere, which, aside from CO_2, may contribute substantially to the greenhouse effect (hence to the behavior of our A in the model). Although the subject is still controversial in the climatology literature, some studies claim in fact that the largest greenhouse effect is in fact that of water vapors in the atmosphere (rather than CO_2). To investigate the possible consequences of these ideas, we decided to introduce the effect of evaporation in the model, and study how this
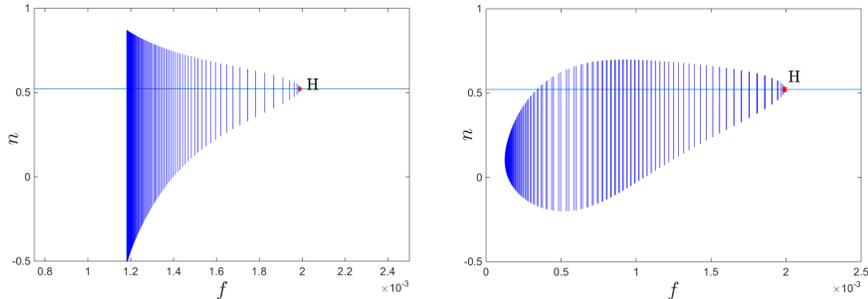
Figure 5: **Supercritical Hopf bifurcation with respect to the parameter** f**. Left.** *The plot shows the equilibrium curve as f changes in the corresponding interval, for fixed $g = 0.001$. The Hopf point (which occurs approximately at $f = 0.002$ is marked on the curve by a red star; we illustrate the evolution of the stable cycle sprouting from the Hopf point, as we allow f and the period to change.* **Right.** *The same equilibrium curve and Hopf point are shown; the stable cycle is extended with respect to both f and g simultaneoulsy. All other system parameters were fixed for this simulation to: $B = 1.9$, $C = 3.04$, $\alpha_w = 0.32$, $\alpha_s = 0.62$, $s_2 = -0.482$, $w_0 = 20$, $T_c = -10$, $R = 0.1266$, $\Omega = 0.474$, $\varepsilon = 0.01264$, $a = 20$.*

changes the model predictions. We increased the negative effect on A at high temperatures, where the water is expected to become vapor and raise up in the atmosphere, contributing to the greenhouse effect, so that

$$\frac{dA}{dt} = \left[1 - g(w - w_0)^2\right] e^{-f(aw - w_0)^2} \tag{9}$$

where a governs the asymmetry in the graph due to vapor bias at high temperatures: more water vapor leads to less OLR and the negative rate of change will be larger in a range of very high temperatures where evaporation occurs. For temperatures even beyond that range, the graph decays to zero, as before. Figure 6 shows a few examples of curves dA/dt for a few representative values of f, g and a, illustrating the effect that each has on the shape of the graph. This shape is not only more appropriate in the assumptions that it makes, and in incorporating a physical condition relevant to the model, but it also renders more plausible systemic behavior, as we show next (see Figure 6 for an illustration of possible dynamic behaviors).

In our simulations, we again observed the effects of changing the sensitivity parameters f and g, for various values of a chosen so that the evaporation range (and the corresponding local minimum) remain within plausible bounds. The effects are now closer to what one would expect from a system of such complexity.

First, aside from stable equilibria, the system exhibits locally stable limit cycles for certain parameter ranges, and may have other stable invariant sets which we have not yet investigated. A locally stable cycle is very significant, since it can stabilize long-term behavior (for initial conditions in a certain attraction basin) to oscillations between low and high temperatures/iceline states. These could be fast oscillations (if the cycle has small period), or long oscillations, on the time-scale of ice ages (if the period is large).
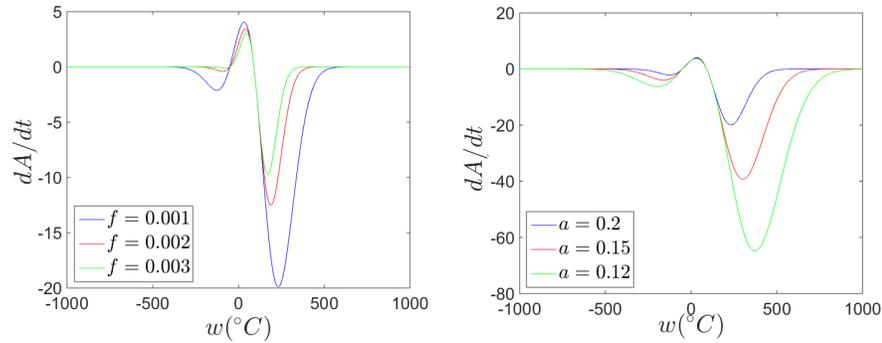
Figure 6: **Graphs of** dA/dt **(with water vapor included) with respect to** w**.** *On the left we see different values of f with fixed $g = 0.001$ and $a = 0.2$. On the right we see different values of a with fixed $f = 0.001$ and $g = 0.001$*

In addition, we found very interesting behavior specifically within a range of parameters compatible with the physical interpretation of our model (see Figure 7). For a fixed, "critical" value of $f \sim 0.002$, the system crosses a *critical* Hopf bifurcation when g is varied, so that a locally attracting equilibrium loses stability with formation of infinitely many non-isolated cycles. In this situation, the system performs the particular oscillation that is dictated by the initial conditions, so that even small changes in the initial state can push trajectories along a different oscillatory pattern (with smaller or larger amplitude and period).

It is believed that complex natural systems invest a lot of effort to achieve function in a critical range (close to a bifurcation state) where these types of transitions are facilitated, allowing the system to converge to different outcomes depending on the initial conditions, and to respond swiftly to environmental changes. Typically, such perfect (critical) transitions are very fragile, and can be destroyed by parameter perturbations. However, in our case, the behavior of the system remains quantitatively (if not qualitatively) similar to the critical behavior under perturbations of f. This is due to the fact that, as f changes in an interval around the critical value $f = 0.002$, the Hopf point remains close to a codimension two generalized Hopf bifurcation. Hence perturbations of f maintain the system close enough to the critical state for the behavior to be almost indistinguishable from critical, at least over periods of a few cycles (i.e., hundreds of thousands of years).

Discussion

Our paper shows how small changes in the parameters of our climatic system may lead to dramatic variations in its long term dynamics. We focused in particular on parameters that described in our model the impact on the climate of known human activity (interpreted from the way in which they affected the level of green house gases produced).

For example, one can interpret the critical transition described in Figure 7 as a possible prediction of systemic behavior when increasing the sensitivity of human activity to temperature (by increasing the value of g). Many view our climatic system
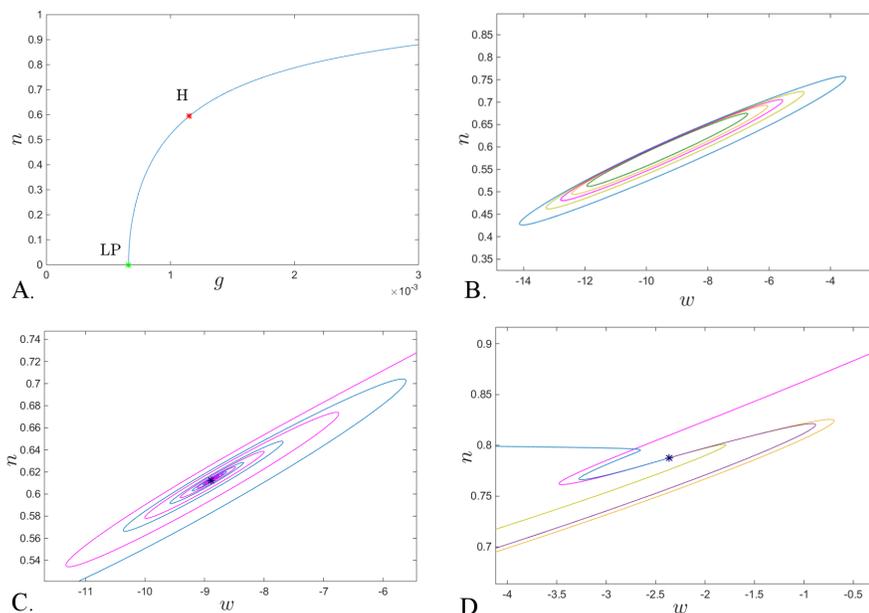
Figure 7: **Critical Hopf-fold bifurcation with respect to the parameter**
g. A. *For a plausible choice of parameter values (see below), the equilibrium*
curve with respect to g has a critical Hopf bifurcation (marked as H) at approx-
imately $g = 0.0011512$, with the birth of infinitely many non-isolated cycles.
The following panels show phase-plane behavior, in a (w, n) projection of the
three-dimensional phase space, for different values of g, on both sides of the
Hopf bifurcation: **B.** *For $g = 0.0011511$, situated to the left of the Hopf bi-*
furcation, there are infinitely many neutral cycles (shown in different colors),
corresponding to different initial conditions. The cycles have large periods (e.g.,
the largest cycle shown in the panel has period 98, measured in thousands of
years, as per our unit conversion). **C.** *For $g = 0.0012$, at the right of the Hopf*
bifurcation, there is a locally stable spiral (shown as a blue star). We show
two different trajectories with initial conditions in the attraction basin. **D.** *For*
$g = 0.002$, further to the right of the Hopf point, the equilibrium becomes an
stable degenerate node (shown as a blue star). We show, in different colors, a
few trajectories converging to this node. For all panels, all other system param-
eters were fixed for this simulation to: $B = 1.9$, $C = 3.04$, $\alpha_w = 0.32$, $\alpha_s = 0.62$,
$s_2 = -0.482$, $w_0 = 20$, $T_c = -10$, $R = 0.1266$, $\Omega = 0.474$, $\varepsilon = 0.01264$, $\alpha = 0.008$,
$f = 0.002$.

as currently operating in a fragile regime where even small state perturbations may
push the system into a different (faster, or wider) oscillatory pattern. However,
oscillations from snow-free to snow-ball Earth are currently expected (and faithfully
rendered by our model) to occur over periods of hundreds of thousands of years.
According to our model, an increase in the human sensitivity g may lead to faster-

spiraling trajectories. While fast damped oscillations may be dramatic in the short term, they are not necessarily bad in the long range, since the spiral eventually leads to a stable equilibrium. However, the position of this equilibrium in the phase space is predicted to change with increasing g, climbing into a range of higher and higher surface average temperatures and ice-lines, compatible to the intuitive eventual effects of global warming: a permanently hot and ice-free Earth.

We may conclude that even subtle changes in our patterns (e.g., exercising moderation – or alternative, non-polluting resources – when using heating/cooling according to seasonal variations, or in response to major cold or hot waves) may lead to a better climatic prognosis. This may be harder in a climate which is already progressing towards dramatic swings (e.g., of temperature, or precipitation) over short periods of time, which may induce people to use more gas-producing resources in order to maintain their current life style. Hence we find it crucial to point out how important it is to try to stick with more conservative patterns of traditional energy consumption. Let us notice, based on our predictions of climate evolution along this degradation path, that factors which would originally help the system regain convergence to less dramatic oscillations may in fact be precisely the factors leading it along an ultimately doomed path.

Climate change is an important subject in the field of mathematical modeling. The Budyko system had been studied a lot over the past few decades, and we believe that there is much more to be done. It is an over simplified model of Earth's climate, but studying it can help us understand the essential behavior of our climate, and how our patterns affect it.

Altogether, it is extremely important to understand, mathematically and philosophically, how the term of "global warming" (defined as the effects of greenhouse gases on the Earth's climate) does not necessarily imply only a steady increase in the planet's annual average temperature (concept still unfortunately used by some scientist to deny these effects, see e.g. [2]), but can in fact refer to much more complex phenomena. Wild cycling between extreme phenomena such as hot and cold periods (from hotter summer and colder winters, to hotter decades and colder decades, etc) may lead to seemingly steady average behavior, without describing the essence of the climatic dynamics.

There is a great amount of work yet to be done on the model, on aspects such as: (1) further refine the model of dA/dt, fine tune parameters to integrate the behavior within realistic functional ranges and, if possible, validate it with empirical measurements; (2) interpret more carefully the contribution of each parameter, trying to segregate the aspects that can be controlled by human behavior from the aspects that cannot be altered, but only understood.

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