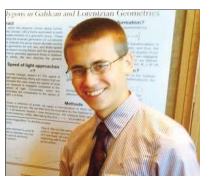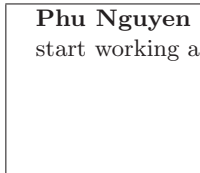# Invariants and Moving Frames for Polygons in Galilean and Lorentzian Geometries

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Abstract We study discrete polygons in Lorentzian and Galilean Klein geometries. We determine explicitly the group-based discrete moving frames for polygons in Galilean and Lorentzian geometries for one, two, and three spatial dimensions. Using these, we find a generating set of independent invariants associated to polygons in these geometries, and we show that given fixed initial conditions, the discrete moving frames and the invariants in Lorentzian geometry approach those in Galilean geometry as the speed of light approaches infinity in the general case.

Introduction

Moving frames had their origins in the mid-19th century with the Frenet-Serret moving frame for continuous curves. Elie Cartan later expanded this concept to group-based moving frames for smooth manifolds, and intense research has been conducted

on such frames. Recent developments in mathematics in the last decade have allowed us to study moving frames for discrete curves or polygons made up of non-connected points.

Applied algebra and group-based moving frames of curves and smooth surfaces have already become ubiquitous in applications from physics to computer vision. However, empirical data are simply collections of points, not continuous objects. Continuous curves usually function as models simply because we have the tools to analyze them. Well-known techniques, such as calculus, are available to study continuous mathematical objects. However, corresponding rigorous techniques still need to be developed in order to analyze discrete mathematical objects.

In our analysis, we utilize the methods of discrete moving frames. A moving frame is a frame of reference in which the observer moves along a curve, surface, or even time. A discrete moving frame is a similar concept, with a frame associated to each vertex of a polygon. The frame of reference is usually described by an element of the geometric group. A new subclass of discrete group-based moving frames are associated to polygons embedded in Klein geometries. These discrete frames provide an algorithm for computing the invariant generators (or curvatures) for these polygons.

In this paper, we study discrete moving frames and their invariants for polygons in Galilean and Lorentzian Klein geometries for one, two, and three spatial dimensions. We interpret the resulting invariants geometrically and show that the invariant matrix of polygons in Lorentzian geometry approaches its Galilean counterpart as the constant, c, approaches infinity.

Notation and Definitions

Polygons and background geometric manifolds

Definition 1. A polygon on a manifold M, $\{\mathbf{x}_n\}_{n=-\infty}^{\infty}$, is a sequence of points with $\mathbf{x}_n \in M$. We say the polygon is a *closed N-gon* if $\mathbf{x}_{n+N} = \mathbf{x}_n$ for all n. Assume a group G is acting on M via the action $\mathbf{x} \mapsto g \cdot \mathbf{x}$, $g \in G$. We say the polygon is *twisted* if there exists $g \in G$ such that $\mathbf{x}_{n+N} = g \cdot \mathbf{x}_n$ for all n. The element g is called the *monodromy* of the polygon, and a closed polygon is a twisted polygon with the monodromy equal the identity.

As a point of notation, if $\mathbf{x}_n \in \mathbb{R}^k$, we let

$$\mathbf{x}_n = \begin{pmatrix} \beta_n \\ \vec{b}_n \end{pmatrix}$$

with $\beta_n \in \mathbb{R}$ and $\vec{b}_n \in \mathbb{R}^{k-1}$ where k is the total number of dimensions (spatial + 1 dimension of time).

Also crucial to any discussion of Galilean and Lorentzian Klein geometries are the following groups:

Definition 2. The *Lorentz group*, $\mathcal{L}_k$, is the set of all $\Theta \in GL_k(\mathbb{R})$ satisfying $\Theta^T J \Theta = J$ for the Minkowski matrix

$$J = \begin{pmatrix} c^2 & \vec{0}^T \\ \vec{0} & -I_{k-1} \end{pmatrix}. \tag{1}$$

The Lorentz group preserves inner products and the degenerate metric defined by J. The Poincaré group is the semidirect product of $\mathcal{L}_k$ and $\mathbb{R}^k$, that is, $\mathfrak{L}_k = \mathcal{L}_k \ltimes \mathbb{R}^k$.

The Poincaré group can be represented by the matrix group

$$\mathfrak{L}_k = \left\{ \begin{pmatrix} 1 & 0 \\ \vec{v} & \Theta \end{pmatrix} : \vec{v} \in \mathbb{R}^k, \, \Theta \in \mathcal{L}_k \right\}, \tag{2}$$

a subgroup of $\mathrm{GL}_{k+1}(\mathbb{R})$.

Definition 3. The Galilean group can also be represented by the subgroup of $\mathrm{GL}_{k+1}(\mathbb{R})$ defined by the matrices

$$\mathfrak{G}_k = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \vec{c} & \vec{a} & \Theta \end{pmatrix} : \alpha \in \mathbb{R}, \, \vec{c}, \vec{a} \in \mathbb{R}^{k-1}, \, \Theta \in SO(k-1) \right\}. \tag{3}$$

We will call G_k the subgroup defined by the lower-right matrix, so that $\mathfrak{G}_k = G_k \ltimes \mathbb{R}^k$ also.

The Galilean and Lorentzian affine geometries are defined to be the Klein geometries [5] associated to the pairs $(\mathfrak{G}_k, G_k)$ and $(\mathfrak{L}_k, \mathcal{L}_k)$, respectively. They are diffeomorphic to $\mathcal{L}_k \ltimes \mathbb{R}^k / \mathcal{L}_k$ and $G_k \ltimes \mathbb{R}^k / G_k$, respectively.

The natural action of $\mathfrak{G}_k$ and $\mathfrak{L}_k$ on these quotients are given by the left multiplication on representatives of the class. If we identify the quotients $\mathcal{L}_k \ltimes \mathbb{R}^k / \mathcal{L}_k$, $G_k \ltimes \mathbb{R}^k / G_k$ (or with the section given by the first column of the matrix), the action is the expected one

$$g = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \vec{c} & \vec{a} & \Theta \end{pmatrix}, \qquad g \cdot \begin{pmatrix} \beta \\ \vec{b} \end{pmatrix} \mapsto \begin{pmatrix} \beta + \alpha \\ \vec{c} + \beta\vec{a} + \Theta\vec{b} \end{pmatrix} \tag{4}$$

for the Galilean action and

$$g = \begin{pmatrix} 1 & 0 \\ \vec{v} & \Theta \end{pmatrix}, \qquad g \cdot \vec{\mathbf{x}} = \Theta\vec{\mathbf{x}} + \vec{v} \tag{5}$$

for the action of the Poincaré group.

Group-based moving frames and discrete moving frames

Group-based moving frames

Given a Lie group G acting on a manifold M with a left action, so that

$$G \times M \to M, \qquad h \cdot (g \cdot z) = (hg) \cdot z,$$

one can define a left (resp. right) group-based moving frame as a map which is equivariant with respect to the action on M and the left (resp. inverse right) action of G on itself, specifically,

$$\rho : M \to G, \qquad \rho(g \cdot z) = g\rho(z) \qquad \left(\text{resp. } \rho(g \cdot z) = \rho(z)g^{-1} \right).$$

We call such an equivariant map a *left (resp. a right) moving frame*. The inverse of a left moving frame is a right one.

Given a group G acting on a manifold M, the existence of a moving frame on the open subset $U \subset M$ is guaranteed if:

(i) the orbits of the group action all have the same dimension and foliate U,

(ii) there is a transverse cross-section $\mathcal{K}$ to the orbits such that for each orbit $\mathcal{O}$, the intersection $\mathcal{O} \cap \mathcal{K}$ contains a single point, and

(iii) the group element taking $z \in \mathcal{O}(z)$ (where $\mathcal{O}(z)$ is the orbit through z) to $\mathcal{O}(z) \cap \mathcal{K}$, is unique.

In this case, a right moving frame $\rho : U \to G$ is given by $\rho(z) \cdot z \in \mathcal{K}$, that is, $\rho(z)$ is the unique element of G taking z to the unique element of $\mathcal{K} \cap \mathcal{O}(z)$. Since $\mathcal{K}$ is transverse to the orbits, the frame defines local coordinates given by $z \mapsto (\rho(z), \rho(z) \cdot z) \in G \times \mathcal{K}$.

In the continuous case of moving frames, the manifold M could be the jet space $J^{(\ell)}(\mathbb{R}^p, M)$. In this case it is known (see [2]) that provided the action is locally effective on subsets, as ℓ grows the prolonged action of G on $J^{(\ell)}(\mathbb{R}^p, M)$ becomes locally free. The work of Boutin (see [1]) discusses what happens for products $M^{\times q}$ as q grows, with G acting with the diagonal action.

A common way to obtain the moving frame is through a normalization process. One can describe normalization equations as those defining the transverse section, $\mathcal{K}$, to the orbits of the group. If the normalization equations are given as $\{\Phi = 0\}$, then the conditions above for the existence of a moving frame are the conditions under which the implicit function theorem can be applied to solve $\Phi(g \cdot z) = 0$ for $g = \rho(z)$. Since both $g = \rho(h \cdot z)$ and $g = \rho(z)h^{-1}$ solve $\Phi(g \cdot (h \cdot z)) = 0$, and the implicit function guarantees a unique solution, then $\rho(h \cdot z) = \rho(z)h^{-1}$, that is, ρ is right-equivariant.

Given a moving frame (left or right) one can generate all possible invariants of the action. Indeed, if ρ is a right moving frame, the function

$$u \mapsto \rho(u) \cdot u$$

is invariant for any $u \in M$; its coordinates are called the *basic invariants*. One can easily see that any invariant of the action is a function of these, using the *replacement rule*: If $I : M \to \mathbb{R}$ is invariant under the action, so that $I(g \cdot u) = I(u)$ for all $g \in G$, then setting $g = \rho(u)$, one obtains

$$I(\rho(u) \cdot u) = I(u).$$

Different choices of the manifold M give rise to different familiar cases. For example, if M is the jet space $J^{(\infty)}(\mathbb{R}^p, P)$ for some manifold P where G acts, and G acts on M via the natural *prolonged action* given by the chain rule, then ρ would generate moving frames on p-submanifolds and the invariants will be standard differential invariants (for example, curvatures, torsions, etc). If $M = \times_k P$ is the Cartesian product of a manifold P where G acts, and G acts on M through the diagonal action, then the invariants are the so-called *joint invariants* (see [4]).

Discrete moving frames

The authors of [3] defined discrete moving frames along polygons, essentially a choice of group element associated to each vertex in an equivariant way.

Let G^N denote the Cartesian product of N copies of the group G. Allow G to act on the left on G^N using the diagonal action $g \cdot (g_r) = (gg_r)$. We also consider what we have called the "right inverse diagonal action" $g \cdot (g_r) = (g_r g^{-1})$.

Definition 4 (Discrete moving frame). We say a map $\rho : M^N \to G^N$ is a left (resp. right) *discrete moving frame* if ρ is equivariant with respect to the diagonal action of G on M^N and the left (resp. right inverse) diagonal action of G on G^N.

Equivalently, a discrete moving frame is a collection of N moving frames on M^N for the diagonal action of G on M^N. The construction of moving frames via transverse

cross-sections can be applied to construct ρ_s. Since $\rho((x_r)) \in G^N$, we will denote by ρ_n its nth component, that is $\rho = (\rho_n)$, where $\rho_n((x_r)) \in G$ for all n. Equivariance means

$$\rho_n(g \cdot (x_r)) = \rho_n((g \cdot x_r)) = g\rho_n((x_r)) \quad (\text{resp. } \rho_n((x_r))g^{-1})$$

for every n. Clearly, if $\rho = (\rho_n)$ is a left moving frame, then $\widehat{\rho} = (\rho_n^{-1})$ is a right moving frame.

As in the original group-based moving frame definition, if $(u_n) \in M^N$, the function $I_n^r : M^N \to M$ is defined by

$$I_n^r(u_1, ..., u_N) = \rho_n(u_1, ..., u_N) \cdot u_r.$$

For any fixed n, the coordinates of the I_n^r with $1 \leq r \leq N$ are a generating set (see [3]). We note that the action induces an action on the coordinate functions, the same as it induces an action on any function, specifically, $g \cdot f(u_r) = f(g \cdot u_r)$. The components of I_n^r will be invariants as I_n^r is an invariant. Those components are called the *basic invariants*.

One can describe a smaller set of generating invariants, the Maurer–Cartan invariants.

Definition 5. Let (ρ_n) be a left (resp. right) discrete moving frame evaluated along a twisted N-gon. The element

$$K_n = \rho_n^{-1}\rho_{n+1} \quad (\text{resp. } \rho_{n+1}\rho_n^{-1})$$

is called the *left (resp. right) Maurer–Cartan invariant matrix for ρ*. We call the equation $\rho_{n+1} = \rho_n K_n$ the discrete left Serret–Frenet equation.

If $G \subset \mathrm{GL}(k, \mathbb{R})$, then the entries of the Maurer–Cartan matrices, together with the basic invariants of I_n^n, generate all other invariants. See [3] for more details.

Explicit moving frames and invariants

Moving Frames and Invariants for Galilean Geometry

Consider the Galilean group with $\mathfrak{G}_k \subset \mathrm{GL}(k + 1)$. Let β be the time component and $\vec{b}$ be the spatial dimension.

For any element $g \in \mathfrak{G}_k$, we have

$$g = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \vec{c} & \vec{a} & \Theta \end{pmatrix}$$

with $\alpha \in \mathbb{R}$, $\vec{c}, \vec{a} \in \mathbb{R}^m$. The action of g on $\mathbf{w} = \begin{pmatrix} \beta \\ \vec{b} \end{pmatrix}$ was determined to be

$$g \cdot \begin{pmatrix} \beta \\ \vec{b} \end{pmatrix} = \begin{pmatrix} \beta + \alpha \\ \vec{c} + \beta\vec{a} + \Theta\vec{b} \end{pmatrix}. \tag{6}$$

Now, denote our polygon by $\{x_n\}_{n=1}^N = \left\{ \begin{pmatrix} \beta_n \\ \vec{b}_n \end{pmatrix} \right\}_{n=1}^N$. Since $\rho_n \in G$, we see

$$\rho_n = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_n & 1 & 0 \\ \vec{c}_n & \vec{a}_n & \Theta_n \end{pmatrix}.$$

Introduce the transverse section defined by the normalization equations for $n = 2$:

$$\rho_n \cdot \begin{pmatrix} \beta_n \\ \vec{b}_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vec{0} \end{pmatrix}, \qquad \rho_n \cdot \begin{pmatrix} \beta_{n+1} \\ \vec{b}_{n+1} \end{pmatrix} = \begin{pmatrix} * \\ \vec{0} \end{pmatrix},$$

where the star $*$ indicates that no normalization condition is placed on that entry. These equations become

$$\begin{pmatrix} 0 \\ \vec{0} \end{pmatrix} = \rho_n \cdot \begin{pmatrix} \beta_n \\ \vec{b}_n \end{pmatrix} = \begin{pmatrix} \beta_n + \alpha_n \\ \vec{c}_n + \beta_n \vec{a}_n + \Theta_n \vec{b}_n \end{pmatrix}$$

$$\begin{pmatrix} * \\ \vec{0} \end{pmatrix} = \rho_n \cdot \begin{pmatrix} \beta_{n+1} \\ \vec{b}_{n+1} \end{pmatrix} = \begin{pmatrix} \beta_{n+1} + \alpha_n \\ \vec{c}_n + \beta_{n+1} \vec{a}_n + \Theta_n \vec{b}_{n+1} \end{pmatrix}$$

so that the following conditions must hold:

$$\beta_n + \alpha_n = 0 \tag{7}$$

$$\vec{c}_n + \beta_n \vec{a}_n + \Theta_n \vec{b}_n = \vec{0} \tag{8}$$

$$\vec{c}_n + \beta_{n+1} \vec{a}_n + \Theta_n \vec{b}_{n+1} = \vec{0}. \tag{9}$$

By (7), we must have $\alpha_n = -\beta_n$. By subtracting (8) from (9), we find that $\Delta\beta_n \vec{a}_n + \Theta_n \Delta\vec{b}_n = 0$, and therefore, as long as $\Delta\beta_n \neq 0$, $\vec{a}_n = -\Theta_n \frac{\Delta\vec{b}_n}{\Delta\beta_n}$ (here, we use the notation $\Delta\beta_n = \beta_{n+1} - \beta_n$). Note that the condition $\Delta\beta_n \neq 0$ implies that two consecutive points on the polygon cannot have the same time component. In other words, there cannot exist two consecutive points at the same time. This is a very natural condition, and we will assume it from now on. Next, using the solution for $\vec{a}_n$ and substituting into equations (8) and (9), we find

$$\alpha_n = -\beta_n$$

$$\vec{a}_n = -\Theta_n \frac{\Delta\vec{b}_n}{\Delta\beta_n}$$

$$\vec{c}_n = \Theta_n \left(\beta_n \frac{\Delta\vec{b}_n}{\Delta\beta_n} - \vec{b}_n \right)$$

which determine the moving frame completely for $k = 2$. The left Maurer–Cartan invariant matrix *for any dimension* is

$$\mathcal{K}_n = \rho_n \rho_{n+1}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_n & 1 & 0 \\ \vec{c}_n & \vec{a}_n & \Theta_n \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\alpha_{n+1} & 1 & 0 \\ \Theta_{n+1}^{-1}(-\vec{c}_{n+1} + \alpha_{n+1}\vec{a}_{n+1}) & -\Theta_{n+1}^{-1}\vec{a}_{n+1} & \Theta_{n+1}^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \Delta\beta_n & 1 & 0 \\ 0 & \Theta_n \Delta\left(\frac{\Delta\vec{b}_n}{\Delta\beta_n}\right) & \Theta_n \Theta_{n+1}^{-1} \end{pmatrix} \tag{10}$$

with different Θ_n for different dimensions.

The case $k = 2$.

If $\Theta_n \in SO(1)$, $\Theta_n = 1$. We conclude that

$$\alpha_n = -\beta_n$$

$$\vec{a}_n = -\frac{\Delta\vec{b}_n}{\Delta\beta_n}$$

$$\vec{c}_n = \beta_n \frac{\Delta \vec{b}_n}{\Delta \beta_n} - \vec{b}_n$$

and the right moving frame is

$$\rho_n = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_n & 1 & 0 \\ c_n & a_n & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\beta_n & 1 & 0 \\ \beta_n \frac{\Delta \vec{b}_n}{\Delta \beta_n} - b_n & -\frac{\Delta \vec{b}_n}{\Delta \beta_n} & 1 \end{pmatrix}.$$

Observe that the left moving frame is simply

$$\rho_n^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \beta_n & 1 & 0 \\ \vec{b}_n & \frac{\Delta \vec{b}_n}{\Delta \beta_n} & 1 \end{pmatrix}$$

which has a more physical interpretation. Finally, consider the calculation of the left Maurer–Cartan matrix $\mathcal{K}_n$

$$\mathcal{K}_n = \rho_n \rho_{n+1}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\beta_n & 1 & 0 \\ \beta_n \frac{\Delta \vec{b}_n}{\Delta \beta_n} - \vec{b}_n & -\frac{\Delta \vec{b}_n}{\Delta \beta_n} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \beta_{n+1} & 1 & 0 \\ \vec{b}_{n+1} & \frac{\Delta \vec{b}_{n+1}}{\Delta \beta_{n+1}} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \Delta \beta_n & 1 & 0 \\ 0 & \Delta \left(\frac{\Delta \vec{b}_n}{\Delta \beta_n} \right) & 1 \end{pmatrix}$$

where $\Delta \vec{b}_{n+1} = \vec{b}_{n+2} - \vec{b}_{n+1}$.

Interpreting this result from a physical point of view, any invariant is generated by the change in time between points in the polygon ($\Delta \beta_n$), and the total change in velocity $\Delta \left(\frac{\Delta \vec{b}_n}{\Delta \beta_n} \right)$ (from it, the discrete acceleration between points x_n and x_{n+1} can be found, which is $\frac{1}{\Delta \beta_n} \Delta \left(\frac{\Delta \vec{b}_n}{\Delta \beta_n} \right)$), and the expression $\frac{\Delta \vec{b}_n}{\Delta \beta_n}$ represents the discrete velocity).

The case $k = 3$.

Here, $\Theta_n \in SO(2)$, so we can assume that

$$\Theta_n = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

for some $\theta \in \mathbb{R}$. To define a cross section we will impose the following additional normalization condition to those of $k = 2$,

$$\rho_n \cdot \begin{pmatrix} \beta_{n+2} \\ \vec{b}_{n+2} \end{pmatrix} = \begin{pmatrix} r_0 \\ r_1 \\ 0 \end{pmatrix}.$$

From here, we get as before the equations in (7-8-9), and to this we add

$$\begin{pmatrix} r_1 \\ 0 \end{pmatrix} = \vec{c}_n + \beta_{n+2} \vec{a}_n + \Theta_n \vec{b}_{n+2} = \Theta_n \left(\Delta_{(2)} \vec{b}_n - \Delta_{(2)} \beta_n \frac{\Delta \vec{b}_n}{\Delta \beta_n} \right),$$

where $\Delta_{(r)} x_n = x_{n+r} - x_n$. Rearranging gives

$$\Theta_n^{-1} \begin{pmatrix} r_1 \\ 0 \end{pmatrix} = \Delta_{(2)} \vec{b}_n - \Delta_{(2)} \beta_n \frac{\Delta \vec{b}_n}{\Delta \beta_n}.$$

Therefore

$$\left\| \begin{pmatrix} r_1 \\ 0 \end{pmatrix} \right\| = r_1 = \left\| \Delta_{(2)} \vec{b}_n - \Delta_{(2)} \beta_n \frac{\Delta \vec{b}_n}{\Delta \beta_n} \right\|$$

where $r_1 > 0$. The first column of Θ_n^{-1} is thus given by

$$\Theta_n^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\Delta_{(2)} \vec{b}_n - \Delta_{(2)} \beta_n \frac{\Delta \vec{b}_n}{\Delta \beta_n}}{\left\| \Delta_{(2)} \vec{b}_n - \Delta_{(2)} \beta_n \frac{\Delta \vec{b}_n}{\Delta \beta_n} \right\|}.$$

This first column determines Θ completely since $\Theta_n^{-1} = \Theta_n^T = (\Psi_n, S\Psi_n)$ where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{11}$$

and

$$\Psi_n = \frac{\Delta_{(2)} \vec{b}_n - \Delta_{(2)} \beta_n \frac{\Delta \vec{b}_n}{\Delta \beta_n}}{\left\| \Delta_{(2)} \vec{b}_n - \Delta_{(2)} \beta_n \frac{\Delta \vec{b}_n}{\Delta \beta_n} \right\|} = \frac{\Delta \vec{b}_{n+1} - \Delta \beta_{n+1} \frac{\Delta \vec{b}_n}{\Delta \beta_n}}{\left\| \Delta \vec{b}_{n+1} - \Delta \beta_{n+1} \frac{\Delta \vec{b}_n}{\Delta \beta_n} \right\|}$$

$$= \frac{\Delta \beta_{n+1} \Delta \left(\frac{\Delta \vec{b}_n}{\Delta \beta_n} \right)}{\left\| \Delta \beta_{n+1} \Delta \left(\frac{\Delta \vec{b}_n}{\Delta \beta_n} \right) \right\|} = \pm \frac{\Delta \left(\frac{\Delta \vec{b}_n}{\Delta \beta_n} \right)}{\left\| \Delta \left(\frac{\Delta \vec{b}_n}{\Delta \beta_n} \right) \right\|}$$

where $\Delta \left(\frac{\Delta \vec{b}_n}{\Delta \beta_n} \right) = \frac{\Delta \vec{b}_{n+1}}{\Delta \beta_{n+1}} - \frac{\Delta \vec{b}_n}{\Delta \beta_n}$. Let us assume from now on that time increases along the polygon and so $\Delta \beta_n > 0$, for all n. The matrix $\Theta_n \in SO(2)$ is just the transpose of Θ_n^{-1}, i.e.,

$$\Theta_n = \begin{pmatrix} \Psi_n^T \\ (S\Psi_n)^T \end{pmatrix},$$

so that

$$\Theta_n \Theta_{n+1}^{-1} = \begin{pmatrix} \Psi_n \cdot \Psi_{n+1} & \Psi \cdot (S\Psi_{n+1}) \\ S\Psi_n \cdot \Psi_{n+1} & \Psi_n \cdot \Psi_{n+1} \end{pmatrix}.$$

From (10), the Maurer-Cartan invariant matrix is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ \Delta \beta_n & 1 & 0 \\ 0 & \Theta_n \Delta \left(\frac{\Delta \vec{b}_n}{\Delta \beta_n} \right) & \Theta_n \Theta_{n+1}^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Delta \beta_n & 1 & 0 \\ 0 & \left\| \Delta \left(\frac{\Delta \vec{b}_n}{\Delta \beta_n} \right) \right\| & \Theta_n \Theta_{n+1}^{-1} \\ & 0 & \end{pmatrix}$$

with $\Theta_n \Theta_{n+1}^{-1}$ as above, which is completely determined by $\Psi_n \cdot \Psi_{n+1}$. A straightforward calculation yields the explicit form:

$$\Theta_n \Theta_{n+1}^{-1} = \begin{pmatrix} \dfrac{\left(\Delta \left(\frac{\Delta \vec{b}_n}{\Delta \beta_n} \right) \right) \cdot \left(\Delta \left(\frac{\Delta \vec{b}_{n+1}}{\Delta \beta_{n+1}} \right) \right)}{\left\| \Delta \left(\frac{\Delta \vec{b}_n}{\Delta \beta_n} \right) \right\| \left\| \Delta \left(\frac{\Delta \vec{b}_{n+1}}{\Delta \beta_{n+1}} \right) \right\|} & \dfrac{-D}{\left\| \Delta \left(\frac{\Delta \vec{b}_n}{\Delta \beta_n} \right) \right\| \left\| \Delta \left(\frac{\Delta \vec{b}_{n+1}}{\Delta \beta_{n+1}} \right) \right\|} \\[4ex] \dfrac{D}{\left\| \Delta \left(\frac{\Delta \vec{b}_n}{\Delta \beta_n} \right) \right\| \left\| \Delta \left(\frac{\Delta \vec{b}_{n+1}}{\Delta \beta_{n+1}} \right) \right\|} & \dfrac{\left(\Delta \left(\frac{\Delta \vec{b}_n}{\Delta \beta_n} \right) \right) \cdot \left(\Delta \left(\frac{\Delta \vec{b}_{n+1}}{\Delta \beta_{n+1}} \right) \right)}{\left\| \Delta \left(\frac{\Delta \vec{b}_n}{\Delta \beta_n} \right) \right\| \left\| \Delta \left(\frac{\Delta \vec{b}_{n+1}}{\Delta \beta_{n+1}} \right) \right\|} \end{pmatrix}$$

where

$$D = \det \begin{pmatrix} 1 & 1 & 1 \\ \frac{\Delta \vec{b}_n}{\Delta \beta_n} & \frac{\Delta \vec{b}_{n+1}}{\Delta \beta_{n+1}} & \frac{\Delta \vec{b}_{n+2}}{\Delta \beta_{n+2}} \end{pmatrix}.$$

D can be found by the existing invariants

$$\frac{D^2}{\left(\left\|\Delta\left(\frac{\Delta \vec{b}_n}{\Delta \beta_n}\right)\right\| \left\|\Delta\left(\frac{\Delta \vec{b}_{n+1}}{\Delta \beta_{n+1}}\right)\right\|\right)^2} + \cos^2\Gamma_n = 1.$$

It is worth noting that in the computation of this matrix, we made use of the fact that, for any $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^2$:

$$\det\begin{pmatrix} 1 & 1 & 1 \\ \vec{a} & \vec{b} & \vec{c} \end{pmatrix} = (S(\vec{b} - \vec{a})) \cdot (\vec{c} - \vec{b})$$

where S is the symplectic matrix defined above. We can now identify the generators of invariants for Galilean polygons in 3 dimensions. As in the lower dimension, the increment in time $\Delta \beta_n$ is an invariant, together with the change in velocity $\left\|\Delta\left(\frac{\Delta \vec{b}_n}{\Delta \beta_n}\right)\right\|$. The extra invariant added by the extra dimension is

$$\Psi_n \cdot \Psi_{n+1} = \cos\Gamma_n$$

where Γ_n is the angle between the two vectors $\vec{A}_n = \Delta\left(\frac{\Delta \vec{b}_n}{\Delta \beta_n}\right)$ and $\vec{A}_{n+1}$.

The case $k = 4$.

One starts seeing clearer how to proceed. Here, $\Theta_n \in SO(3)$, which is 3-dimensional, and once again we will impose an additional normalization condition to define the cross-section, namely

$$\rho_n \cdot \begin{pmatrix} \beta_{n+3} \\ \vec{b}_{n+3} \end{pmatrix} = \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ 0 \end{pmatrix}.$$

Then, to the previous equations we add

$$\begin{pmatrix} r_1 \\ r_2 \\ 0 \end{pmatrix} = \vec{c}_n + \beta_{n+3}\vec{a}_n + \Theta_n \vec{b}_{n+3} = \Theta_n\left(\Delta_{(3)}\vec{b}_n - \Delta_{(3)}\beta_n \frac{\Delta \vec{b}_n}{\Delta \beta_n}\right).$$

Rearranging gives

$$\Theta_n^{-1}\begin{pmatrix} r_1 \\ r_2 \\ 0 \end{pmatrix} = \Delta_{(3)}\vec{b}_n - \Delta_{(3)}\beta_n \frac{\Delta \vec{b}_n}{\Delta \beta_n}.$$

Let us define

$$\phi_{n,r} = \Delta_{(r)}\vec{b}_n - \Delta_{(r)}\beta_n \frac{\Delta \vec{b}_n}{\Delta \beta_n}.$$

As before, $\phi_{n,2} = \Delta\beta_{n+1}\vec{A}_n$. One can equally show that

$$\phi_{n,3} = (\Delta\beta_{n+1} + \Delta\beta_{n+2})\vec{A}_n + \Delta\beta_{n+2}\vec{A}_{n+1},$$

and

$$\phi_{n,3} = \Theta_n^{-1}\begin{pmatrix} r_1 \\ 0 \\ 0 \end{pmatrix} + \Theta_n^{-1}\begin{pmatrix} 0 \\ r_2 \\ 0 \end{pmatrix}.$$

From calculations similar to the case $k = 3$, we know that the first column of Θ_n^{-1} is

$$\Theta_n^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\phi_{n,2}}{\|\phi_{n,2}\|} = \pm \frac{\vec{A}_n}{\|\vec{A}_n\|} = \Psi_{n,1}.$$

Then,

$$\phi_{n,3} - r_1 \frac{\vec{A}_n}{\|\vec{A}_n\|} = \Theta_n^{-1} \begin{pmatrix} 0 \\ r_2 \\ 0 \end{pmatrix}$$

and from here, we see that

$$\pm \left\| \phi_{n,3} - r_1 \frac{\vec{A}_n}{\|\vec{A}_n\|} \right\| = r_2 \qquad \text{and} \qquad \frac{\phi_{n,3} - r_1 \frac{\phi_{n,2}}{\|\phi_{n,2}\|}}{r_2} \cdot \frac{\vec{A}_n}{\|\vec{A}_n\|} = 0.$$

Solving the second equation, we get

$$r_1 = \|\phi_{n,2}\| \frac{\phi_{n,3} \cdot \vec{A}_n}{\phi_{n,2} \cdot \vec{A}_n} = \frac{1}{\|\vec{A}_n\|} \left((\Delta\beta_{n+1} + \Delta\beta_{n+2}) \|\vec{A}_n\|^2 + \Delta\beta_{n+2} \vec{A}_n \cdot \vec{A}_{n+1} \right),$$

Notice that both r_1 and r_2 are generated by $\Delta\beta_r$, $\|\vec{A}_r\|$ and $\vec{A}_r \cdot \vec{A}_{r+1}$, for all r. We then get the second column of Θ_n^{-1} by considering

$$\Psi_{n,2} = \frac{\phi_{n,3} - r_1 \frac{\phi_{n,2}}{\|\phi_{n,2}\|}}{\|\phi_{n,3} - r_1 \frac{\phi_{n,2}}{\|\phi_{n,2}\|}\|} = \Theta_n^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The third column is just the cross product of the first two. We have

$$\Theta_n^{-1} = (\Psi_{n,1}, \Psi_{n,2}, \Psi_{n,1} \times \Psi_{n,2}).$$

Since $\Theta_n \in SO(3)$, it is just the transpose of Θ_n^{-1}, and hence

$$\begin{pmatrix} \Psi_{n,1} \cdot \Psi_{n+1,1} & \Psi_{n,1} \cdot \Psi_{n+1,2} & \Psi_{n,1} \cdot (\Psi_{n+1,1} \times \Psi_{n+1,2}) \\ \Psi_{n,2} \cdot \Psi_{n+1,1} & \Psi_{n,2} \cdot \Psi_{n+1,2} & \Psi_{n,2} \cdot (\Psi_{n+1,1} \times \Psi_{n+1,2}) \\ (\Psi_{n,1} \times \Psi_{n,2}) \cdot \Psi_{n+1,1} & (\Psi_{n,1} \times \Psi_{n,2}) \cdot \Psi_{n+1,2} & (\Psi_{n,1} \times \Psi_{n,2}) \cdot (\Psi_{n+1,1} \times \Psi_{n+1,2}) \end{pmatrix}.$$

We can calculate the Maurer–Cartan invariant matrix as in (10). In this case

$$\Theta_n \Delta \left(\frac{\Delta b_n}{\Delta \beta_n} \right) = \Theta \vec{A}_n = \begin{pmatrix} \|\vec{A}_n\| \\ 0 \\ 0 \end{pmatrix}$$

and $\Theta_n \Theta_{n+1}^{-1}$ is as above.

As we can see, as the dimension goes up we gain an additional invariant which is independent from its lower-dimensional analogues. In this case all entries we know are generated by the known invariants: $\Delta\beta_n$ (time change), $\|\vec{A}_n\|^2$ (curvature) and $\vec{A}_n \cdot \vec{A}_{n+1}$ (angle), for all n. We need to find the extra invariant that the fourth dimension is contributing. It clearly appears in $\Psi_{n,1} \cdot (\Psi_{n+1,1} \times \Psi_{n+1,2})$, for example. This entry is given by

$$\frac{\Delta\beta_{n+3}}{\|\vec{A}_n\| \|\vec{A}_{n+1}\| \|\phi_{n+1,3} - r_1^{n+1} \phi_{n+1,2}\|} \vec{A}_n \cdot \left(\vec{A}_{n+1} \times \vec{A}_{n+2} \right).$$

Therefore, the third independent invariant is the volume created by the three vectors $\vec{A}_n$, $\vec{A}_{n+1}$, $\vec{A}_{n+2}$. We can call it *the discrete Galilean torsion*.

Moving Frames and Invariants for Lorentzian Geometry

As in the previous subsection, we will construct a right moving frame by applying normalization conditions that will determine it uniquely.

Let $\rho_n = \begin{pmatrix} 1 & 0 \\ \vec{v}_n & \Theta_n \end{pmatrix}$, $\vec{v}_n \in \mathbb{R}^k$, $\Theta_n \in \mathcal{L}_k$ and $J = \begin{pmatrix} c^2 & 0 \\ 0 & -I \end{pmatrix}$. For any dimension, we begin with the constraint

$$\rho_n \cdot x_n = \mathbf{0}. \tag{12}$$

Then, we have $\rho_n \cdot x_n = \vec{v}_n + \Theta_n x_n = \mathbf{0}$, and so $\vec{v}_n = -\Theta_n x_n$. Therefore, the right moving frame for the Lorentzian case is $\rho_n = \begin{pmatrix} 1 & 0 \\ -\Theta_n x_n & \Theta_n \end{pmatrix}$, and the left moving frame is

$$\rho_n^{-1} = \begin{pmatrix} 1 & 0 \\ x_n & \Theta_n^{-1} \end{pmatrix},$$

for some Θ_n to be determined by additional normalization conditions. The general left Maurer–Cartan invariant matrix is given by

$$\mathcal{K}_n = \rho_n \rho_{n+1}^{-1} = \begin{pmatrix} 1 & 0 \\ -\Theta_n x_n & \Theta_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x_{n+1} & \Theta_{n+1}^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \Theta_n \Delta x_n & \Theta_n \Theta_{n+1}^{-1} \end{pmatrix}. \tag{13}$$

The case $k = 2$.

We impose the additional condition

$$\rho_n \cdot x_{n+1} = \vec{v}_n + \Theta_n x_{n+1} = \begin{pmatrix} r_1 \\ 0 \end{pmatrix}. \tag{14}$$

By substitution, $\Theta_n \Delta x_n = \begin{pmatrix} r_1 \\ 0 \end{pmatrix}$ and $\Delta x_n = \Theta_n^{-1} \begin{pmatrix} r_1 \\ 0 \end{pmatrix}$. Recall that $\Theta_n^{-1} \in \mathcal{L}_n$, so

$$\left\| \Theta_n^{-1} \begin{pmatrix} r_1 \\ 0 \end{pmatrix} \right\|_J = \left\| \begin{pmatrix} r_1 \\ 0 \end{pmatrix} \right\|_J = cr_1,$$

and from here

$$r_1 = \frac{1}{c} \|\Delta x_n\|_J, \quad \text{and} \quad \Theta_n^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{c\Delta x_n}{\|\Delta x_n\|_J}$$

where $\|\Delta x_n\|_J > 0$. Now, recall the general form of an element of the group $\mathcal{L} = \{\Omega \in GL_2(\mathbb{R}) : \Omega^T J \Omega = J\}$ for $J = \begin{pmatrix} c^2 & 0 \\ 0 & -1 \end{pmatrix}$, assuming that $\det \Omega = 1$. It is given by matrices of the form

$$\Omega = \begin{pmatrix} \cosh \xi & \frac{1}{c} \sinh \xi \\ c \sinh \xi & \cosh \xi \end{pmatrix} = (\mu_1, R\mu_1)$$

where

$$R = \begin{pmatrix} 0 & \frac{1}{c^2} \\ 1 & 0 \end{pmatrix}.$$

Since $\Theta_n^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{c\Delta x_n}{\|\Delta x_n\|_J}$ and $\Theta_n^{-1} \in \mathcal{L}$, we conclude that

$$\Theta_n^{-1} = \begin{pmatrix} \frac{c\Delta x_n}{\|\Delta x_n\|_J} & R\frac{c\Delta x_n}{\|\Delta x_n\|_J} \end{pmatrix}, \quad \text{and} \quad \Theta_n = J^{-1}\Theta^{-T}J.$$

Let us denote by $T_n = \frac{c\Delta x_n}{\|\Delta x_n\|_J}$ and $N_n = R\frac{c\Delta x_n}{\|\Delta x_n\|_J}$, the tangent and the normal at the vertex x_n. Then, the Maurer–Cartan matrix is as in (13) with

$$\Theta_n \Delta x_n = \begin{pmatrix} \frac{1}{c}\|\Delta x_n\|_J \\ 0 \end{pmatrix}, \quad \text{and} \quad \Theta_n \Theta_{n+1}^{-1} = \begin{pmatrix} \frac{1}{c^2}\langle T_n, T_{n+1}\rangle_J & \frac{1}{c^2}\langle T_n, N_{n+1}\rangle_J \\ -\langle N_n, T_{n+1}\rangle_J & -\langle N_n, N_{n+1}\rangle_J \end{pmatrix}.$$

The first two invariants are $\frac{\Delta x_n}{\|\Delta x_n\|_J}$ and $\cosh\xi = \langle T_n, T_{n+1}\rangle_J$ (the hyperbolic angle).

Proposition 6. *As $c \to +\infty$, the moving frame ρ_n and the Maurer–Cartan matrix $\mathcal{K}_n$ converge to their Galilean counterpart.*

Proof. Let $x_n = \begin{pmatrix} \beta_n \\ \vec{b}_n \end{pmatrix}$. For convenience, note that

$$\frac{c\Delta x_n}{\|\Delta x_n\|_J} = \frac{c\Delta x_n}{\sqrt{c^2\Delta\beta_n^2 - \Delta\vec{b}_n^2}} = \frac{\Delta x_n/\Delta\beta_n}{\sqrt{1 - \frac{\Delta\vec{b}_n^2}{c^2\Delta\beta_n^2}}} = \frac{1}{\sqrt{1 - \frac{\Delta\vec{b}_n^2}{c^2\Delta\beta_n^2}}} \begin{pmatrix} 1 \\ \frac{\Delta\vec{b}_n}{\Delta\beta_n} \end{pmatrix}.$$

Define $\gamma_n = \frac{1}{\sqrt{1 - \frac{\Delta\vec{b}_n^2}{c^2\Delta\beta_n^2}}}$, so that $\gamma_n \to 1$ as $c \to +\infty$. Then

$$\Theta_n^{-1} = \begin{pmatrix} \gamma_n & \gamma_n\frac{\Delta\vec{b}_n}{c^2\Delta\beta_n} \\ \gamma_n\frac{\Delta\vec{b}_n}{\Delta\beta_n} & \gamma_n \end{pmatrix} \quad \text{and} \quad \Theta_n = J^{-1}\Theta_n^{-T}J = \begin{pmatrix} \gamma_n & -\gamma_n\frac{\Delta\vec{b}_n}{c^2\Delta\beta_n} \\ -\gamma_n\frac{\Delta\vec{b}_n}{\Delta\beta_n} & \gamma_n \end{pmatrix}.$$

Therefore, the left moving frame is given by

$$\rho_n^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \beta_n & \gamma_n & \gamma_n\frac{\Delta\vec{b}_n}{c^2\Delta\beta_n} \\ \vec{b}_n & \gamma_n\frac{\Delta\vec{b}_n}{\Delta\beta_n} & \gamma_n \end{pmatrix}$$

and as $c \to \infty$, the moving frame approaches their Galilean counterpart

$$\rho_n^{-1} \to \begin{pmatrix} 1 & 0 & 0 \\ \beta_n & 1 & 0 \\ \vec{b}_n & \frac{\Delta\vec{b}_n}{\Delta\beta_n} & 1 \end{pmatrix}.$$

The limit of $\mathcal{K}_n$ follows from $\mathcal{K}_n = \rho_n\rho_{n+1}^{-1}$. $\qquad\qquad\square$

The case $k = 3$.

Again, using (12) we immediately have $v_n = -\Theta_n x_n$, and (14) becomes

$$\Theta_n x_{n+1} + v_n = \Theta_n \Delta x_n = \begin{pmatrix} r_1^n \\ 0 \\ 0 \end{pmatrix},$$

that is, $r_1^n = \frac{1}{c}\|\Delta x_n\|_J$ and $\Theta_n^{-1}e_1 = c\frac{\Delta x_n}{\|\Delta x_n\|_J} = T_n$, where e_i is the standard basis of $\mathbb{R}^3$ with a 1 in place i and zero elsewhere.

We impose a second normalization given by

$$\Theta_n x_{n+2} + v_n = \Theta_n \Delta_{(2)} x_n = \Theta_n \Delta x_{n+1} + \Theta_n \Delta x_n = \begin{pmatrix} r_2^n \\ r_3^n \\ 0 \end{pmatrix}.$$

Rearranging the second equation gives

$$\Theta_n \Delta x_{n+1} = \begin{pmatrix} r_2^n - r_1^n \\ r_3^n \\ 0 \end{pmatrix}.$$

Since $\Theta_n \in \mathcal{L}$, the Lorentzian metric must be preserved, and

$$\|\Theta_n \Delta x_{n+1}\|_J = \left\| \begin{pmatrix} r_2^n - r_1^n \\ r_3^n \\ 0 \end{pmatrix} \right\|_J = \sqrt{c^2 (r_2^n - r_1^n)^2 - (r_3^n)^2},$$

and $r_3^n = \sqrt{c^2(r_2^n - r_1^n)^2 - \|\Delta x_{n+1}\|_J^2}$. Furthermore, we must also have

$$\left\langle \Theta_n \Delta x_n, \Theta_n \Delta_{(2)} x_n \right\rangle_J = c^2 r_1^n r_2^n = \left\langle \Delta x_n, \Delta_{(2)} x_n \right\rangle_J.$$

Therefore

$$r_2^n = \frac{\left\langle \Delta x_n, \Delta_{(2)} x_n \right\rangle_J}{c^2} \frac{1}{r_1^n} = \frac{\left\langle \Delta x_n, \Delta_{(2)} x_n \right\rangle_J}{c \|\Delta x_n\|_J} = \frac{\|\Delta x_n\|_J}{c} + \frac{\left\langle \Delta x_n, \Delta x_{n+1} \right\rangle_J}{c \|\Delta x_n\|_J}.$$

Finally, we recall that

$$\begin{aligned}
r_3^n &= \sqrt{c^2(r_2^n - r_1^n)^2 - \|\Delta x_{n+1}\|_J^2} \\
&= \sqrt{\frac{\left\langle \Delta x_n, \Delta x_{n+1} \right\rangle_J^2}{\|\Delta x_n\|_J^2} - \|\Delta x_{n+1}\|_J^2} \\
&= \|\Delta x_{n+1}\|_J \sqrt{\left(\frac{\left\langle \Delta x_n, \Delta x_{n+1} \right\rangle_J}{\|\Delta x_n\|_J \|\Delta x_{n+1}\|_J} \right)^2 - 1}.
\end{aligned}$$

This expression suggests the definition

$$\cosh \varphi_n = \frac{\left\langle \Delta x_n, \Delta x_{n+1} \right\rangle_J}{\|\Delta x_n\|_J \|\Delta x_{n+1}\|_J}$$

where φ_n has the geometrical interpretation of being the hyperbolic angle between Δx_n and Δx_{n+1}, as in Figure 1.
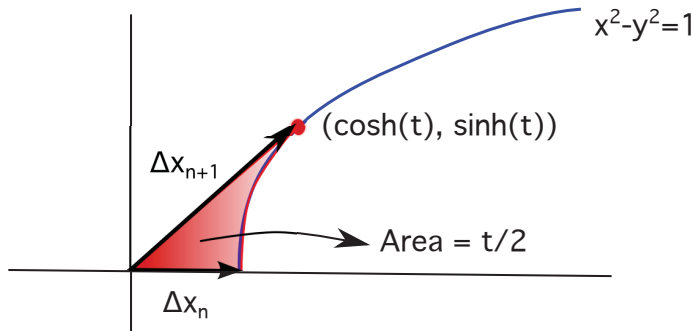

Figure 1: The geometric interpretation of hyperbolic angles.

Recall that if $\Omega = (\xi_1, \xi_2, \xi_3)$, $\Omega \in \mathcal{L}$ and $\det \Omega > 0$, then $\xi_3 = \xi_1 \times_{\mathcal{L}} \xi_2$, where $\times_{\mathcal{L}}$ is the *Lorentzian cross product* given by

$$\xi_1 \times_{\mathcal{L}} \xi_2 = \begin{pmatrix} -\frac{1}{c^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xi_1 \times \xi_2 = -J^{-1} \xi_1 \times \xi_2. \tag{15}$$

Indeed, one can directly check the formula

$$\|\xi_1 \times_{\mathcal{L}} \xi_2\|_J^2 = \frac{1}{c^2} \left(\|\xi_1\|_J^2 \|\xi_2\|_J^2 - \langle \xi_1, \xi_2 \rangle_J^2 \right)$$

which shows that $\|\xi_1 \times_{\mathcal{L}} \xi_2\|_J^2 = -1$ whenever $\|\xi_1\|_J^2 = c^2$ and $\|\xi_2\|_J^2 = -1$. Also, $\det \Omega = -\|\xi_1 \times_{\mathcal{L}} \xi_2\|_J^2 = 1$.

Let $\xi_{n,i}$ denote the ith column vector of Θ_n^{-1}. Note that, as before,

$$\Theta_n^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \xi_{n,1} = \frac{c \Delta x_n}{\|\Delta x_n\|_J} \tag{16}$$

and

$$\Theta_n^{-1} \begin{pmatrix} r_2^n - r_1^n \\ r_3^n \\ 0 \end{pmatrix} = (r_2^n - r_1^n)\xi_{n,1} + r_3^n \xi_{n,2} = \Delta x_{n+1}.$$

Solving for $\xi_{n,2}$ yields

$$\xi_{n,2} = \frac{1}{\sinh \varphi_n} \left(\frac{\Delta x_{n+1}}{\|\Delta x_{n+1}\|_J} - \cosh \varphi_n \frac{\Delta x_n}{\|\Delta x_n\|_J} \right). \tag{17}$$

Finally, using the Lorentzian cross product (15)

$$\xi_{n,3} = -J^{-1} \frac{c(\Delta x_n \times \Delta x_{n+1})}{\sinh \varphi_n \|\Delta x_n\|_J \|\Delta x_{n+1}\|_J}.$$

Now, we turn to the computation of the Maurer–Cartan matrix. Using $\Theta_n = J^{-1} \Theta_n^{-T} J$, we compute

$$\Theta_n \Theta_{n+1}^{-1} = J^{-1} \begin{pmatrix} \xi_{n,1}^T \\ \xi_{n,2}^T \\ \xi_{n,3}^T \end{pmatrix} J \begin{pmatrix} \xi_{n+1,1} & \xi_{n+1,2} & \xi_{n+1,3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{c^2} \langle \xi_{n,1}, \xi_{n+1,1} \rangle_J & \frac{1}{c^2} \langle \xi_{n,1}, \xi_{n+1,2} \rangle_J & \frac{1}{c^2} \langle \xi_{n,1}, \xi_{n+1,3} \rangle_J \\ -\langle \xi_{n,2}, \xi_{n+1,1} \rangle_J & -\langle \xi_{n,2}, \xi_{n+1,2} \rangle_J & -\langle \xi_{n,2}, \xi_{n+1,3} \rangle_J \\ -\langle \xi_{n,3}, \xi_{n+1,1} \rangle_J & -\langle \xi_{n,3}, \xi_{n+1,2} \rangle_J & -\langle \xi_{n,3}, \xi_{n+1,3} \rangle_J \end{pmatrix}.$$

To ease the notation, denote by σ_n the hyperbolic angle $\cosh \sigma_n = \frac{\langle \Delta x_n, \Delta x_{n+2} \rangle_J}{\|\Delta x_n\|_J \|\Delta x_{n+2}\|_J}$, and let $V = \det \begin{pmatrix} \frac{\Delta x_n}{\|\Delta x_n\|_J} & \frac{\Delta x_{n+1}}{\|\Delta x_{n+1}\|_J} & \frac{\Delta x_{n+2}}{\|\Delta x_{n+2}\|_J} \end{pmatrix}$.

Straightforward calculations show that $\Theta_n \Theta_{n+1}^{-1}$ is given by

$$\begin{pmatrix} \cosh \varphi_n & \dfrac{\cosh \sigma_n - (\cosh \varphi_n)(\cosh \varphi_{n+1})}{c \sinh \varphi_{n+1}} & \dfrac{-V}{(\sinh \varphi_{n+1})} \\[2em] c \sinh \varphi_n & \dfrac{(\cosh \varphi_n)(\cosh \sigma_n) - (\cosh \varphi_n)^2(\cosh \varphi_{n+1})}{(\sinh \varphi_n)(\sinh \varphi_{n+1})} & \dfrac{-c \cosh \varphi_n V}{(\sinh \varphi_n)(\sinh \varphi_{n+1})} \\[2em] 0 & \dfrac{cV}{(\sinh \varphi_n)(\sinh \varphi_{n+1})} & X \end{pmatrix}$$

where

$$X = -\frac{c^2(\Delta x_n \times \Delta x_{n+1})^T J^{-1}(\Delta x_{n+1} \times \Delta x_{n+2})}{\|\Delta x_n\|_J \|\Delta x_{n+1}\|_J^2 \|\Delta x_{n+2}\|_J \sinh \varphi_n \sinh \varphi_{n+1}}.$$

As before, the hyperbolic angles, φ, σ, have a natural geometric interpretation. By definition,

$$\cosh \varphi_n = \left\langle \frac{\Delta x_n}{\|\Delta x_n\|}, \frac{\Delta x_{n+1}}{\|\Delta x_{n+1}\|} \right\rangle_J.$$

As in Figure 1, in two dimensions, this corresponds to twice the area enclosed by the hyperbola $c^2 t^2 - y^2 = 1$ and the unit vectors $\frac{\Delta x_n}{\|\Delta x_n\|}$ and $\frac{\Delta x_{n+1}}{\|\Delta x_{n+1}\|}$. In three dimensions, this same interpretation is valid after applying an element of the Lorentzian group to bring the two vectors into the x,t-plane. With three dimensions, there are three invariants: $\frac{\Delta x_n}{\|\Delta x_n\|}$ (Lorentzian vector), $\cosh \varphi_n$ (hyperbolic angle), and V (Euclidean volume).

Proposition 7. *If $c \to +\infty$, the Lorentzian moving frame and the Maurer–Cartan matrix converge to their Galilean counterparts.*

Proof. As before, to prove this statement, and since $\mathcal{K}_n = \rho_n \rho_{n+1}^{-1}$, we only need to show that ρ_n converges to its Galilean counterpart as $c \to +\infty$.

Recall that

$$\rho_n = \begin{pmatrix} 1 & 0 \\ -\Theta_n^{-1} \vec{v}_n & \Theta_n^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x_n & \Theta_n^{-1} \end{pmatrix}.$$

Therefore, we only need to show that the three columns of $\Theta_n^{-1} = (\xi_{1,n}, \xi_{2,n}, \xi_{3,n})$, converge to those in the Galilean case, namely

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{\Delta \vec{b}_n}{\Delta \beta_n} & \Psi_n & S\Psi_n \end{pmatrix}$$

with $\Psi_n = \dfrac{\Delta\left(\frac{\Delta \vec{b}_n}{\Delta \beta_n}\right)}{\left\|\Delta\left(\frac{\Delta \vec{b}_n}{\Delta \beta_n}\right)\right\|}$ and S as in (11).

First of all, notice that $\frac{1}{c} \|\Delta x_n\|_J = \sqrt{(\Delta \beta_n)^2 - \frac{1}{c^2} \left\|\Delta \vec{b}_n\right\|_J^2} \xrightarrow[c \to +\infty]{} \Delta \beta_n$. From (16) we have

$$\xi_{1,n} = \frac{c \Delta x_n}{\|\Delta x_n\|_J} \xrightarrow[c \to \infty]{} \begin{pmatrix} 1 \\ \frac{\Delta \vec{b}_n}{\Delta \beta_n} \end{pmatrix}.$$

Now, notice that

$$c \sinh \varphi_n = c\sqrt{\left(\frac{\langle \Delta x_n, \Delta x_{n+1}\rangle_J}{\|\Delta x_n\|_J \|\Delta x_{n+1}\|_J}\right)^2 - 1}$$

$$= \sqrt{c^2\left(\frac{\langle \Delta x_n, \Delta x_{n+1}\rangle_J^2 - \|\Delta x_n\|_J^2 \|\Delta x_{n+1}\|_J^2}{\|\Delta x_n\|_J^2 \|\Delta x_{n+1}\|_J^2}\right)}$$

$$= \sqrt{\frac{(\|\Delta \vec{b}_n \Delta \beta_{n+1} - \Delta \vec{b}_{n+1}\Delta \beta_n\|^2) + \frac{1}{c^2}((\Delta \vec{b}_n \cdot \Delta \vec{b}_{n+1})^2 - \|\Delta \vec{b}_n\|^2\|\Delta \vec{b}_{n+1}\|^2)}{(\Delta \beta_n^2 - \frac{1}{c^2}\|\Delta \vec{b}_n\|^2)(\Delta \beta_{n+1}^2 - \frac{1}{c^2}\|\Delta \vec{b}_{n+1}\|^2)}}$$

$$\xrightarrow[c \to \infty]{} \sqrt{\frac{\|\Delta \vec{b}_n \Delta \beta_{n+1} - \Delta \vec{b}_{n+1}\Delta \beta_n\|^2}{\Delta \beta_n^2 \Delta \beta_{n+1}^2}} = \sqrt{\left\|\frac{\Delta \vec{b}_n}{\Delta \beta_n} - \frac{\Delta \vec{b}_{n+1}}{\Delta \beta_{n+1}}\right\|^2} = \left\|\Delta\left(\frac{\Delta \vec{b}_n}{\Delta \beta_n}\right)\right\|,$$

while

$$\cosh\varphi_n = \frac{\langle\Delta x_n, \Delta x_{n+1}\rangle_J}{\|\Delta x_n\|_J\|\Delta x_{n+1}\|_J} = \frac{\Delta\beta_n\Delta\beta_{n+1} - \frac{1}{c^2}\Delta\vec{b}_n\cdot\Delta\vec{b}_{n+1}}{\sqrt{((\Delta\beta_n)^2 - \frac{1}{c^2}\|\Delta\vec{b}_n\|^2)((\Delta\beta_{n+1})^2 - \frac{1}{c^2}\|\Delta\vec{b}_{n+1}\|^2)}}$$

$$\underset{c\to\infty}{\longrightarrow} \frac{\Delta\beta_n\Delta\beta_{n+1}}{\sqrt{(\Delta\beta_n)^2(\Delta\beta_{n+1})^2}} = 1.$$

Therefore,

$$\xi_{2,n} = \frac{1}{c\sinh\varphi_n}\left(\frac{c\Delta x_{n+1}}{\|\Delta x_{n+1}\|_J} - \cosh\varphi_n\frac{c\Delta x_n}{\|\Delta x_n\|_J}\right) \underset{c\to\infty}{\longrightarrow} \frac{1}{\left\|\Delta\left(\frac{\Delta\vec{b}_n}{\Delta\beta_n}\right)\right\|}\left(\Delta\left(\frac{0}{\frac{\Delta\vec{b}_n}{\Delta\beta_n}}\right)\right).$$

Finally, $\xi_{3,n} = -J^{-1}\xi_{1,n}\times\xi_{2,n}$ and so, since $-J^{-1}\to\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}$, in the limit

$$\xi_{3,n} \underset{c\to\infty}{\longrightarrow} \left(0, S\frac{\Delta\left(\frac{\Delta\vec{b}_n}{\Delta\beta_n}\right)}{\left\|\Delta\left(\frac{\Delta\vec{b}_n}{\Delta\beta_n}\right)\right\|}\right)^T = (0, S\Psi)^T$$

concluding the proof since we can now see that the Lorentzian moving frame exactly approaches the Galilean case as $c\to\infty$. $\qquad\square$

Limits in the General Case

Consider the general invariant matrix for the Lorentzian case,

$$\mathcal{K}_n = \begin{pmatrix} 1 & 0 \\ \Theta_n\Delta x_n & \Theta_n\Theta_{n+1}^{-1} \end{pmatrix}.$$

In an effort to consider the case as $c\to\infty$, we now derive a useful decomposition for a matrix in the Lorentzian rotation group. First, take $\Omega\in GL_n(\mathbb{R})$, such that $\Omega^T J'\Omega = J'$ for $J' = \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}$. Furthermore, let $\Omega = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$ where $D\in\mathbb{R}$, B is $(n-1)\times 1$, C is $1\times(n-1)$, and A is $(n-1)\times(n-1)$. Then

$$\Omega^T J\Omega = \begin{pmatrix} D & B^T \\ C^T & A^T \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} D & C \\ B & A \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}.$$

Let $B = -D\sigma$ for $\sigma\in\mathbb{R}^n$. Then, we have the following system of equations [6]

$$A^T A - C^T C = I$$

$$B^T A - DC = 0$$

$$B^T B - D^2 = -1.$$

Solving yields the solutions

$$D = \pm\frac{1}{\sqrt{1-\|\sigma\|^2}}$$

$$AA^T = I_n - D^2\sigma\sigma^T$$

$$A = A^T = \pm\left(I + \frac{D-1}{\|\sigma\|^2}\sigma\sigma^T\right)E$$

$$C = -\sigma^T A$$

where $E \in SO(k-1)$. We then get

$$\Omega = \begin{pmatrix} D & \mp D\sigma^T \\ -D\sigma & \pm\left(I + \frac{D-1}{\|\sigma\|^2}\sigma\sigma^T\right) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}.$$

However, for $\Theta_n \in \mathcal{L}_n$ and

$$J = \begin{pmatrix} c^2 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \frac{1}{c^2} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} c^2 & 0 \\ 0 & I \end{pmatrix},$$

we have $\Theta^T J \Theta = J$. Let

$$N = \begin{pmatrix} c^2 & 0 \\ 0 & I \end{pmatrix},$$

so that $NJN^{-1} = J'$. Then, for Ω satisfying $\Omega^T J'\Omega = J'$, $(N^{-1}\Omega^T N)J(N^{-1}\Omega N) = J$ and therefore (since N is symmetric) $(N^{-1}\Omega N)^T J(N^{-1}\Omega N) = J$. This mapping is bijective; for any $\Theta \in \mathcal{L}$, there exists Ω satisfying the above properties such that $N^{-1}\Omega N = \Theta$. Therefore, for any $\Theta \in \mathcal{L}$, we can write

$$\Theta = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} D & \mp D\sigma^T \\ -D\sigma & \pm\left(I + \frac{D-1}{\|\sigma\|^2}\sigma\sigma^T\right) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & I \end{pmatrix}.$$

Thus, for any $\Theta \in \mathcal{L}$, we can decompose the matrix as

$$\Theta = \begin{pmatrix} D & \frac{1}{c}\mp D\sigma^T \\ -cD\sigma & \pm\left(I + \frac{D-1}{\|\sigma\|^2}\sigma\sigma^T\right) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}$$

where $D \in \mathbb{R}$, $E \in SO(k-1)$ and $\sigma \in \mathbb{R}^{k-1}$. We will now use this result to demonstrate that the Lorentzian moving frames subject to the given normalization conditions approach that of the Galilean case. Clearly, as $c \to \infty$, the condition $\rho_n x_n = \mathbf{0}$ approaches the appropriate Galilean condition. Each subsequent normalization condition in the Lorentzian case corresponds to a constraint

$$v_n + \Theta^{\mathcal{L}} x_{n+i} = \Theta^{\mathcal{L}} \Delta_{(i-1)} x_n = \begin{pmatrix} \vec{u} \\ \vec{0} \end{pmatrix}$$

where the superscript indicates that $\Theta^{\mathcal{L}} \in \mathcal{L}$ and $\vec{u} \in \mathbb{R}^k$ for the $(k-1)^{th}$ condition. By the second such condition ($\vec{u} \in \mathbb{R}$), we find

$$\Theta^{-1}\begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} = \begin{pmatrix} \gamma_n \\ \gamma_n \frac{\vec{b}_n}{\beta_n} \end{pmatrix}.$$

Therefore, if we restrict our attention to the case of the proper Lorentz transformations, then by the above decomposition with the appropriate sign conventions,

$$\Theta^{-1} = \begin{pmatrix} \gamma_n & \frac{1}{c^2}\gamma_n\left(\frac{\Delta\vec{b}_n}{\beta_n}\right)^T \\ -\gamma_n\left(\frac{\Delta\vec{b}_n}{\beta_n}\right) & \left(I + \frac{\gamma_n-1}{\|\left(\frac{\Delta\vec{b}_n}{\beta_n}\right)\|^2}\left(\frac{\Delta\vec{b}_n}{\beta_n}\right)\left(\frac{\Delta\vec{b}_n}{\beta_n}\right)^T\right) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}$$

remembering that $\gamma_n = \dfrac{1}{\sqrt{1 - \frac{\Delta \vec{b}_n^2}{c^2 \Delta \beta_n^2}}}$. We then have

$$
\lim_{c \to \infty} \Theta^{\mathcal{L}} \Delta x_n = \left[\lim_{c \to \infty} \Theta^{-1} \right]^{-1} \left(\begin{array}{c} \Delta_{(n-1)} \beta_n \\ \Delta_{(n-1)} \vec{b}_n \end{array} \right)
$$

$$
= \left[\left(\begin{array}{cc} 1 & 0 \\ -\left(\frac{\Delta \vec{b}_n}{\beta_n} \right) & I \end{array} \right) \left(\begin{array}{cc} 1 & 0 \\ 0 & E \end{array} \right) \right]^{-1} \left(\begin{array}{c} \Delta_{(n-1)} \beta_n \\ \Delta_{(n-1)} \vec{b}_n \end{array} \right)
$$

$$
= \left(\begin{array}{cc} 1 & 0 \\ -E^{-1}\left(\frac{\Delta \vec{b}_n}{\beta_n} \right) & E^{-1} \end{array} \right) \left(\begin{array}{c} \Delta_{(n-1)} \beta_n \\ \Delta_{(n-1)} \vec{b}_n \end{array} \right)
$$

$$
= \left(\begin{array}{cc} 1 & 0 \\ 0 & E^{-1} \end{array} \right) \left(\begin{array}{c} \Delta_{(n-1)} \beta_n \\ \Delta_{(n-1)} \beta_n \left(\frac{\Delta_{(n-1)} \vec{b}_n}{\Delta_{(n-1)} \beta_n} - \frac{\Delta \vec{b}_n}{\Delta \beta_n} \right) \end{array} \right) = \left(\begin{array}{c} \vec{u}' \\ \vec{0} \end{array} \right).
$$

Recalling the definitions put forth in the section on Galilean invariants, we find

$$
\left(\begin{array}{c} \Delta_{(n-1)} \beta_n \\ E^{-1} \Psi_{n-1} \end{array} \right) = \left(\begin{array}{c} u_1 \\ \tilde{u} \\ \vec{0} \end{array} \right).
$$

Remember that $\Psi_{n-1} = \Delta_{(n-1)} \beta_n \left(\frac{\Delta_{(n-1)} \vec{b}_n}{\Delta_{(n-1)} \beta_n} - \frac{\Delta \vec{b}_n}{\Delta \beta_n} \right)$, which constrains

$$
E^{-1} \Psi_{n-1} = \left(\begin{array}{c} \tilde{u} \\ \vec{0} \end{array} \right)
$$

for some $\tilde{u} \in \mathbb{R}^{k-2}$. This constraint is the same one placed on the submatrix $\Theta_n \in SO(k-1)$ in the Galilean moving frame. Therefore, the normalization conditions of the Lorentzian case approach those of the Galilean case as $c \to \infty$, as proposed.

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