Transfinite Diameter

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The transfinite diameter is a way of quantifying the size of compact sets in Euclidean space. This quantity is related to the Hausdorff dimension and the Lebesgue measure, but gives a slightly different perspective on the set than either of those do. In this paper, we introduce the transfinite diameter, and outline some attempts to calculate this quantity for three sets in $\mathbb{R}^2$.

For $z_1, z_2, \ldots, z_n \in \mathbb{R}^m$ we define

$$p(z_1, z_2, \ldots, z_n) = \prod_{j>i} |z_j - z_i|.$$

Then, given a compact set $E \subset \mathbb{R}^m$, we define for all integers $n \geq 2$,

$$p_n(E) = \sup \{ p(z_1, z_2, \ldots z_n) \mid z_1, z_2, \ldots, z_n \in E \}.$$

It is readily apparent that for any appropriate set E, we have $p_2(E) = \operatorname{diam} E$.

The first observation that we must make is that the supremum in the definition of $p_n(E)$ is actually the maximum of the given set. That is, there is a set of n points in E on which p achieves its maximum $p_n(E)$. This is true because p is a continuous function on the compact set E^n, the cartesian product of E with itself n times. Therefore, we can consider a set of n points for which $p(z_1^*, z_2^*, \ldots, z_n^*) = p_n(E)$. Now, we let

$$d_n(E) = [p_n(E)]^{\frac{2}{n(n-1)}} = [p_n(E)]^{\frac{1}{\binom{n}{2}}}.$$

The value d_n is the geometric mean of the distances between the $\binom{n}{2}$ pairs of points formed by this set of n points for which p achieves its maximum. Before we can actually define the quantity known as the transfinite diameter, we need the following fact.

Lemma 1. *For any natural number $n \geq 2$ and compact set $E \subset \mathbb{R}^m$,*

$$d_{n+1}(E) \leq d_n(E). \tag{1}$$

That is, $d_2(E), d_3(E), \ldots$ is a decreasing sequence.

The proof of this fact starts with a collection of $n+1$ points $\{x_1^*, x_2^*, \ldots, x_{n+1}^*\}$ for which p_{n+1} achieves its maximum d_{n+1}. Then, dividing out all of the distances that involve x_{n+1}^* gives us the product of all the distances between the points $\{x_1^*, x_2^*, \ldots, x_n^*\}$. This product is bounded above by d_n. Finding this product for each point x_k^* gives us a number of terms which are all bounded above by d_n. Taking the product of all these numbers, after a little manipulation, gives us the required bound on d_{n+1}.

Since the sequence $d_2, d_3, \ldots$ is a monotonic sequence of nonnegative terms, it must converge to a limit. This limit is called the transfinite diameter, and is denoted by $\tau(E)$.

The Unit Circle

Now, for a practical example of the transfinite diameter, we turn to a familiar set in $\mathbb{R}^2$, the unit circle.

Theorem 2. *Let E be the unit circle in $\mathbb{R}^2$. Then $\tau(E) = 1$.*

PROOF. The unit circle is most easily described by considering it in the complex plane, and using Euler's formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

The point $e^{i\theta}$ is the point on the unit circle that is rotated θ radians from standard position. Then, with some trigonometry, we express the distance between two such points based on the difference between the angles that describe them. We then use some multivariable calculus to find that the products of the distances between n points is maximized when the points are equally spaced around the unit circle, in the shape of a regular n-gon.

We consider these points that optimize the function p_n as if they were in the complex plane. Let $z_1 = 1 = e^{i \cdot 0}$. According to our previous calculation, the next point in our collection will be rotated $\frac{2\pi}{n}$ radians from 1. Thus the second point is $e^{i\frac{2\pi}{n}}$. Continuing in this manner, the other points that are equally spaced around the unit circle are the other complex nth roots of unity, and every nth root of unity will be in our collection. So, every point in our collection is a root of the equation

$$x^n - 1 = 0.$$

Thus,

$$x^n - 1 = (x-1)(x - e^{i\frac{2\pi}{n}})(x - e^{i\frac{4\pi}{n}}) \ldots (x - e^{i\frac{2n-2\pi}{n}}).$$

However, we also have

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \ldots + x + 1),$$

and dividing out the $(x-1)$ term yields

$$(x - e^{i\frac{2\pi}{n}})(x - e^{i\frac{4\pi}{n}}) \ldots (x - e^{i\frac{2n-2\pi}{n}}) = (x^{n-1} + x^{n-2} + \ldots + x + 1).$$

Substituting 1 for x in each polynomial gives:

$$(1+1+1+\ldots+1) = n = (1-e^{i\frac{2\pi}{n}})(1-e^{i\frac{4\pi}{n}})\ldots(1-e^{i\frac{2n-2\pi}{n}}).$$

This implies that

$$n = |(1-e^{i\frac{2\pi}{n}})(1-e^{i\frac{4\pi}{n}})\ldots(1-e^{i\frac{2n-2\pi}{n}})| = |(1-e^{i\frac{2\pi}{n}})|\ldots|(1-e^{i\frac{2n-2\pi}{n}})|.$$

This product of the distances from $z_1 = 1$ to each other point in the collection is denoted by $C(z_1)$, what we call the contribution of 1. By the symmetry of the configuration of points, the contribution is the same for each point in the collection. Thus we have

$$\prod_{j=0}^{n-1} C(e^{i\frac{2j\pi}{n}}) = n^n.$$

Given any two points z_j and z_k, the distance between them appears twice in the preceding product, once in $C(z_j)$ and once in $C(z_k)$. Thus we have

$$p_n^2 = n^n$$

$$p_n = n^{\frac{n}{2}}$$

$$p_n^{\frac{2}{n(n-1)}} = d_n = n^{\frac{1}{n-1}}$$

and so

$$\tau(E) = \lim_{n\to\infty} d_n = 1.$$

The Unit Segment

The next set we investigate is the unit segment E = [0,1]. We can consider this as a subset of $\mathbb{R}^2$, the set $\{(x,0)|x \in [0,1]\}$, but the results of all calculations are the same if we consider the segment as a subset of the real line, which we do for simplicity. Now, to investigate the transfinite diameter of this set, we look at the values of p that are created by points that are equally spaced along the segment. For example, $p(0, \frac{1}{2}, 1) = \frac{1}{4}$.

Now, we consider the general set of points $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}\}$. (Note that this set contains $n+1$ points.) The value for p for this set of points can be calculated fairly easily because of the regularity of the set. We will then consider the limit of this value as n goes to infinity to give a lower bound for $\tau([0,1])$, since the points are not actually the optimal points. The product $p(0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1)$ contains $n(n+1)/2$ terms. Each of these terms will be a fraction with denominator n. The numerators of these fractions will contain n values of 1, $n-1$ values of 2, etc., up to 1 value of n. Thus, we have

$$p(0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}) = \frac{n!(n-1)!(n-2)!\ldots 2!1!}{n^{\frac{n(n+1)}{2}}}.$$

Then, we define

$$h_n = p(0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n})^{\frac{2}{(n+1)n}} = \frac{(n!(n-1)!(n-2)!\ldots 2!1!)^{\frac{2}{(n+1)n}}}{n}.$$

Examining the formula for h_n shows that it satisfies the recurrence relation

$$h_{n+1} = \frac{[((h_n n)^{\frac{(n+1)n}{2}})((n+1)!)]^{\frac{2}{(n+2)(n+1)}}}{n+1}$$

which we manipulate to obtain

$$\frac{h_{n+1}^{(n+2)}}{h_n^n} = \frac{n^n}{(n+1)^n} \frac{[(n+1)!]^{\frac{2}{n+1}}}{(n+1)^2}. \tag{2}$$

Now we turn our attention momentarily to h_n itself. The values of h_n all lie in $[0, 1]$, a compact set. So the sequence h_n has a convergent subsequence. Thus, we can assume that $\lim_{n\to\infty} h_n = A$, as long as we only consider the values of h_n along that subsequence. We then take the corresponding limit of both sides of equation (2) to give

$$\frac{A^{n+2}}{A^n} = A^2 = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} \frac{[(n+1)!]^{\frac{2}{n+1}}}{(n+1)^2}.$$

Then, after a good amount of simplification and estimation involving Stirling's approximation for $n!$ [3], we have

$$A = e^{-1/2} e^{-1} \lim_{n\to\infty} (2n\pi)^{\frac{1}{2n+2}} = e^{-3/2} \approx 0.22313.$$

Since, by the definition of d_n, we have $h_n \leq d_{n+1}$, we take the limit as n goes to infinity of both sides to get

$$A \leq \tau([0, 1]) \leq 1.$$

The transfinite diameter of the segment is bounded above by 1, since the distance between any two points must be less than or equal to 1, and the successive d_n values are the geometric means of a number of such distances.

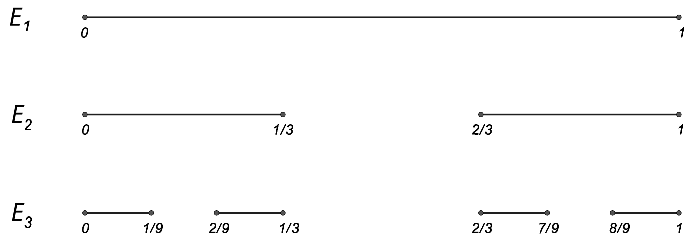
Unlike in the case of the unit circle, this first group of sets of points that we have analyzed does not turn out to give the actual values for d_n, i.e., $p(z_1, z_2, \ldots, z_n)$ is not maximized for $z_k = \frac{k-1}{n}$. Some calculus methods can be used to determine the points that actually maximize $p(z_1, z_2, \ldots, z_n)$. Unfortunately, the numbers produced by these calculations do not seem to have the sort of pattern that would lend itself to a closed form solution in n, nor any easily determined limit behavior. However, the numerical estimations of these numbers are all higher than the corresponding values for h_n, suggesting that the inequality derived before is most likely strict. (And in fact, other approaches to calculating the transfinite diameter show that $\tau([0, 1]) = .25$, thus showing that our inequality is strict.)

Note here that what we have actually done is estimated $\tau([0, 1])$ by only using the subset of points in $B = \{0\} \cup \{\frac{k}{n} | k, n, \in \mathbb{Z}\}$. Again, though, the small change in the limit that is produced when we allow ourselves to use irrational numbers is difficult to calculate precisely once we sacrifice the regularity allowed by using specific sets of rational numbers.

The Cantor Set

The last set we will investigate is the middle-third Cantor set, which is defined in the following way. Let E_1 be $[0,1]$. Then, E_n is obtained from E_{n-1} by removing the middle third of each segment present in E_{n-1}. Then, E_2 is $[0,1/3]\cup[2/3,1]$, etc. We then define E, the middle-third Cantor set, as follows

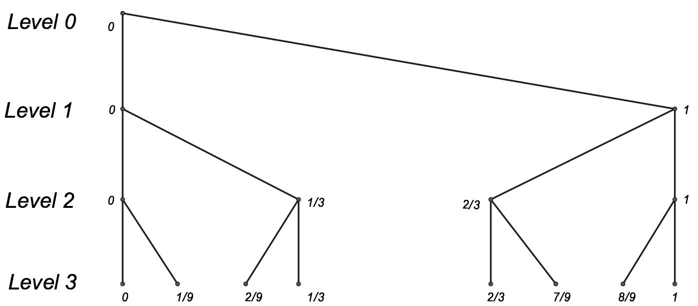
$$E = \bigcap_{n=1}^{\infty} E_n.$$



This set has a number of extremely interesting properties, many of which are rather surprising. First of all, it is clear that any point that is ever the endpoint of an interval in any set E_n is a member of E. What is not obvious is the number of other points that are also members of the set. It turns out that the Cantor set itself is actually uncountable, although it has measure zero. Also, the set is not only closed, it is perfect, meaning it contains all of its limit points and every point in the set is a limit point.

Now, to investigate the transfinite diameter of the set, we turn to what are the most "special" points of the set and therefore a logical first choice for inputs into p. These are the endpoints of the intervals for any given E_k. We notice first of all that E_k is the union of 2^{k-1} disjoint intervals of length $(\frac{1}{3})^{k-1}$. Then E_k will contain 2^k endpoints, including 0 and 1. So, for our first estimate of the value of p_{2^k}, we input the endpoints of the intervals of E_k into our function p. Now, unfortunately, the irregularity of the positions of these endpoints makes it difficult to evaluate the limit behavior of the function p as n increases without bound. However, we can find a relatively easy lower bound for this value.

To do this, we consider the construction of the Cantor set as if it applies only to the endpoints themselves. We start, then, with only a point, which we label 0. We call this set, $\{0\}$, "level 0". We then imagine this point splitting off into two points labelled 0 and 1. Then the set {0,1} is what we call "level 1". Next, we imagine that 0 splits off into 0 and 1/3, and that 1 breaks into 2/3 and 1, so that level 2 is $\{0,1/3,2/3,1\}$. Similarly, 0 becomes 0 and 1/9, 1/3 becomes 2/9 and 1/3, etc., creating level 3, which is the set $\{0,1/9,2/9,3/9,6/9,7/9,8/9,1\}$. This process can be fairly easily visualized as a complete binary tree with the root labelled as 0, its "children" labelled as 0 and 1, etc. Then, we can get at least an estimate of the distance between any two points based on their latest common "ancestor". This approach to this estimate is based on a similar process used in [1].



To see how this works, fix any point x that is an endpoint of an interval in E_n. Now, the point on the opposite end of that interval will have its latest common ancestor with x at the level $n-1$. This will be the only point satisfying this property. Similarly, there will be two points sharing their last common ancestor with x two levels above the nth, at level $n-2$. Because of the tree structure of these ancestor-descendant relationships, for each k from 0 to $n-1$, there are 2^{n-1-k} endpoints in E_n whose last common ancestor with x is at the kth level. We consider $k=0$ to mean that the two points have no common ancestor, so that one of them has 0 as its first ancestor, and the other has 1 as its first ancestor. Also note that this accounts for all of the other 2^n-1 endpoints of E_n.

Now, for each of these categories of points, we find a lower bound for the distance from one of these points to x. If x' has its last common ancestor with x at the level k, then x' and x have different ancestors at level $k+1$. These ancestors, since they have the same parent, are the opposite endpoints of an interval in E_{k+1}. Then, as a result of the division that occurs from E_{k+1} to E_{k+2}, the descendants of these opposite endpoints will be on opposite sides of the middle third of that interval. Thus, the distance between them will be at least $(\frac{1}{3})^{k+1}$, the length of that middle third. Then, if we call A_k the set of endpoints of intervals in E_k, we combine all this information to yield

$$\prod_{x'\neq x;\, x,\, x'\in A_k} |x-x'| \geq \prod_{L=0}^{k-1}((\frac{1}{3})^{L+1})^{2^{k-1-L}}.$$

Now, using this estimate for each of the 2^k points in A_k, we have

$$(p(x_1,x_2,\ldots,x_{2^k}))^2 \geq \left(\prod_{L=0}^{k-1}((\frac{1}{3})^{L+1})^{2^{k-1-L}}\right)^{2^k}.$$

The fact that the p-value is squared in the previous expression arises from the fact that each distance $|x_j-x_m|$ occurs twice in the complete product, once when $x=x_i$, and once when $x=x_j$. So, we raise both sides to the appropriate power to yield

$$(p_{2^k})^{\frac{2}{2^k(2^k-1)}} = d_{2^k} \geq (p(x_1, x_2, \ldots, x_{2^k}))^{\frac{2}{2^k(2^k-1)}} \geq$$

$$\left(\prod_{L=0}^{k-1} ((\frac{1}{3})^{L+1})^{2^{k-1-L}}\right)^{\frac{2^k}{2^k(2^k-1)}}.$$

Simplifying this expression by taking the natural logarithm of both sides to change the infinite product into a series, and eventually taking the limit as k goes to infinity gives

$$1 \geq \tau(E) \geq \frac{1}{9} = 0.11111\ldots.$$

(Again, the transfinite diameter is bounded above by one since the distance between any two points in the set is bounded above by 1.)

We can refine this lower bound by considering separately the one point x' that has its last common ancestor with x at the level $k-1$. Since these points are opposite endpoints of the same interval in E_k, the distance between them is $(\frac{1}{3})^{k-1}$. The next refinement we can introduce is to divide the points x' into two groups for level in our tree structure, i.e., for each value of L.

For example, when $k = 4$ and $L = 0$, there are 8 points that share a common ancestor with x only at level 0. If we assume, in the worst case, that $x = 1/3$, then those 8 points are 2/3, 19/27, 20/27, 7/9, 8/9, 25/27, 26/27, and 1. So, dividing these into two groups, we treat half of these as if they were all 2/3, and the other half as if they were all 8/9. The points such that $L = 1$, which share their last common ancestor with x at level 1, are 0, 1/27, 2/27, and 1/9. Again, we group together 0 and 1/27, and 2/27 and 1/9. Unfortunately, the estimate for the distances between x and these points must be valid for $x = 2/9$ as well as for $x = 1/3$. Thus the best we can say is that $|x - x'| \geq 5/27 = 5(1/3)^3$ for the first pair, and $|x - x'| \geq 1/9 = (1/3)^2$ for the second pair.

Finally, we consider the two points that share their last common ancestor with x at level 2, 2/9 and 7/27. Again, we must produce an estimate that is valid for $x = 8/27$ as well as for $x = 1/3$. Then the best possible estimate assumes only $|x - 2/9| \geq 2/27 > 5/81 = 5(1/3)^4$, and $|x - 7/27| \geq 1/27 = (1/3)^3$ Then, for $x = 1/3$, factoring back in the previously mentioned single point that shares its last common ancestor with x at level $k-1$, we have $\prod_{x' \neq x, x \in E_k} |x - x'| \geq (\frac{1}{3})^4 (\frac{5}{9})^4 (\frac{1}{9})^2 (\frac{5}{27})^2 (\frac{1}{27}) (\frac{5}{81}) (\frac{1}{27})$. This generalizes to:

$$\prod_{x' \neq x; x, x' \in A_k} |x - x'| \geq (\frac{1}{3})^{k-1} \prod_{L=0}^{k-2} ((\frac{1}{3})^{L+1})^{2^{k-2-L}} \prod_{L=0}^{k-2} (5(\frac{1}{3})^{L+2})^{2^{k-2-L}}.$$

Note that this estimate sacrifices a bit by lowering the estimated distance of $6(1/3)^k$, at the level $k-2$, to $5(1/3)^k$, for the sake of clarity and efficiency in the product. (This is why we used 5/81 instead of 2/27 in the $k = 4, L = 2$ example above.) In the limit as k increases without bound, this sacrifice is negligible.

As before, we simplify our expression using the natural logarithm to give a lower bound on $\tau(E)$. Now, this process can be continued in this direction, dividing the points at each level into more and more smaller groups of points. However, the refinements on the estimates that this process creates begin to level off fairly quickly, and the largest lower bound that is created with this process yields $\tau(E) \geq 0.162$.

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