Ball State Undergraduate Mathematics Exchange
https://digitalresearch.bsu.edu/mathexchange
Vol. 17, No. 1 (Fall 2023)
Pages 15 – 33

Classification of seven-dimensional solvable Lie algebras with five-dimensional abelian nilradicaly

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Abstract

This paper provides a classification of seven-dimensional indecomposable solvable Lie algebras over $\mathbb{R}$ for which the nilradical is five-dimensional and abelian. We follow a technique that was first introduced by Mubarakzyanov.

1 Introduction

For the elementary theory of Lie algebras refer to [4, 6, 7]. It has to be understood that classifying solvable Lie algebras is a different exercise from studying the semisimple

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algebras. The problem of classifying all semisimple Lie algebras over the field of complex numbers was solved by Cartan in 1894 [1], and over the field of real numbers by Gantmacher in 1939 [2]. For solvable indecomposable Lie algebras the problem is much more difficult. The classification of solvable Lie algebras only exists for low dimensions and was performed by, amongst others, Mubarakzyanov for solvable Lie algebras of dimension $n \leq 5$ over the field of real and partially over the field of complex numbers in [11] and [12]. Mubarakzyanov's results are summarized in [17]. Mubarakzyanov also considered dimension six and classified solvable Lie algebras with a co-dimension one nilradical [13]. Shabanskaya and Thompson refined his results and found some missing cases in [19, 20]. Then Turkowski classified six-dimensional solvable Lie algebras with a co-dimension two nilradical in [21]. Nilpotent Lie algebras in dimension six were studied as far back as Umlauf [22], and later by Morozov [9].

It is probably impossible to classify solvable Lie algebras in general in arbitrary dimension. The first step in classifying solvable Lie algebras in a specific dimension is to find the possible nilradicals. A general theorem asserts that if $\mathfrak{g}$ is an $n-$dimensional solvable Lie algebra, the dimension of its nilradical $\mathfrak{nil}(\mathfrak{g})$ is at least $\dfrac{n}{2}$ [13]. So for $n = 7$, the possible dimensions of the nilradical are seven, six, five, or four. The seven-dimensional nilradicals, called the nilpotents, were studied by Seely over $\mathbb{R}$ [18] and by Gong over $\mathbb{C}$ [3]. The four-dimensional nilradical case was studied by Hindeleh and Thompson [5]. The six-dimensional nilradical case was studied by Parry [16]. The five-dimensional nilradical case is still an open problem. A complete classification consists of all possible five-dimensional nilpotent algebras, including the decomposable ones. In this paper we study the case where the nilradical is the five-dimensional abelian algebra $\mathbb{R}^5$.

We note that Ndogmo and Winternitz outlined methods for classifying solvable Lie algebras with abelian nilradical for a general dimension in [14, 15]. Also, while this work was being finalized, Le, Vu A, et al. [8] posted in arXiv methods for the classification of seven-dimensional Lie algebras with five-dimensional nilradical. They conclude with the number of possible algebras without explicitly finding them. This paper provides a complete list of the seven-dimensional solvable Lie algebras with a five-dimensional abelian nilradical.

In section , we recall basic definitions and properties related to the classification of solvable Lie algebras. Then in section , we use Turkowski's method [21] for classifying solvable Lie algebras with abelian nilradical, that is also outlined by Ndogmo and Winternitz [14, 15]. Finally, we list the adjoint matrices corresponding to our algebras with trivial and one-dimensional centers in sections 3.1 and 3.2, respectively. The complete list of algebras can be found in tables 1, and 2.

2 A Method to Obtain the Solvable Algebras

2.1 General Concepts

A Lie algebra $\mathfrak{g}$ is *solvable* if its derived series DS terminates, i.e.

$$DS = \{\mathfrak{g}_0 = \mathfrak{g}, \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \ldots, \mathfrak{g}_k = [\mathfrak{g}_{k-1}, \mathfrak{g}_{k-1}] = 0\}$$

for some $k \geq 1$.

A Lie algebra $\mathfrak{g}$ is *nilpotent* if its central series CS terminates, i.e.

$$CS = \{\mathfrak{g}^{(0)} = \mathfrak{g}, \mathfrak{g}^{(1)} = [\mathfrak{g},\mathfrak{g}], \ldots, \mathfrak{g}^{(k)} = [\mathfrak{g}, \mathfrak{g}^{(k-1)}] = 0\}$$

for some $k \geq 1$.

A solvable algebra $\mathfrak{g}$ has a decomposition of the form

$$\mathfrak{g} = \mathfrak{nil}(\mathfrak{g}) \oplus X,$$

satisfying

$$\begin{aligned} [\mathfrak{nil}(\mathfrak{g}), \mathfrak{nil}(\mathfrak{g})] &\subset \mathfrak{nil}(\mathfrak{g}), \\ [\mathfrak{nil}(\mathfrak{g}), X] &\subseteq \mathfrak{nil}(\mathfrak{g}), \\ [X,X] &\subset \mathfrak{nil}(\mathfrak{g}), \end{aligned} \tag{1}$$

where $\mathfrak{nil}(\mathfrak{g})$ denotes the nilradical of $\mathfrak{g}$, the vector space X is spanned by the remaining generators, and $\oplus$ denotes the direct sum of vector spaces.

An element n of $\mathfrak{g}$ is *nilpotent* if it satisfies

$$[\ldots[[x,n],n]\ldots n] = 0$$

for all $x \in \mathfrak{g}$ when the commutator is taken sufficiently many times.

A set of elements $\{x_1, \ldots, x_k\}$ of $\mathfrak{g}$ is called *nilindependent* if no non-trivial linear combination of them is nilpotent.

For $x \in \mathfrak{g}$, the *adjoint transformation* of x is a linear transformation $ad_x : \mathfrak{g} \to \mathfrak{g}$ defined by

$$ad_x(y) = [x,y],$$

for all $y \in \mathfrak{g}$. In this paper, the restriction of ad_x to the nilradical of $\mathfrak{g}$ denoted $ad_x|_{\mathfrak{nil}(\mathfrak{g})}$ is realized by matrices $A \in gl(5,\mathbb{R})$. Notice that if n is a nilpotent element of $\mathfrak{g}$, then $ad_n|_{\mathfrak{nil}(\mathfrak{g})}$ is a nilpotent matrix.

A set of matrices in $gl(n,\mathbb{R})$ will be called *linearly nilindependent* if no non-trivial linear combination of them is nilpotent.

2.2 Basic Structural Theorems

We shall choose a basis for $\mathfrak{g} = \langle e_1, e_2, \ldots, e_5, x_1, x_2\rangle$ where $e_i \in \mathfrak{nil}(\mathfrak{g}), x_\alpha \in X$, for $i = 1, \ldots, 5$, and $\alpha = 1, 2$.

To classify the seven-dimensional solvable Lie algebras with five-dimensional nilradical, one must start with a five-dimensional nilpotent algebra that will form $\mathfrak{nil}(\mathfrak{g})$, and add $X = \langle x_1, x_2\rangle$ satisfying the properties in (5). The following are all the nilpotent Lie algebras up to isomorphism in dimension five: $\mathbb{R}^5$, $A_{3,1} \oplus \mathbb{R}^2$, $A_{4,1} \oplus \mathbb{R}$, and $A_{5,1} - A_{5,6}$, where $\mathbb{R}^n$ denotes the $n-$dimensional abelian algebra, and $A_{n,k}$ denotes the k^{th} algebra of dimension n from Patera's list [17]. The focus of this article is on the first case, namely $\mathfrak{nil}(\mathfrak{g}) = \mathbb{R}^5$.

Since the nilradical is abelian and basis elements must satisfy the relations in (5), we have

$$[e_i, e_j] = 0 \tag{2a}$$

$$\begin{pmatrix} [x_\alpha, e_1] \\ \vdots \\ [x_\alpha, e_5] \end{pmatrix} = \begin{pmatrix} e_1 & \dots & e_5 \end{pmatrix} A^\alpha \tag{2b}$$

$$[x_1, x_2] = R^i e_i \tag{2c}$$

where $A^\alpha = ad_{x_\alpha}|_{\mathfrak{nil}(\mathfrak{g})}$, $\alpha = 1,2$ and $i, j = 1,\dots,5$ and we use the Einstein summation notation. The classification of our Lie algebras thus amounts to classification of the matrices A^α and the constants R^i.

By the Jacobi identity involving x_1, x_2, and an e_i,

$$[[x_1,x_2],e_i] + [[x_2,e_i],x_1] + [[e_i,x_1],x_2] = 0.$$

Thus

$$\begin{aligned} ad_{[x_1,x_2]}(e_i) &= [x_1,[x_2,e_i]] - [x_2,[x_1,e_i]] \\ &= ad_{x_1}([x_2,e_i]) - ad_{x_2}([x_1,e_1]) \\ &= ad_{x_1}(ad_{x_2}(e_i)) - ad_{x_2}(ad_{x_1}(e_i)) \\ &= [ad_{x_1}, ad_{x_2}](e_i). \end{aligned}$$

Hence $[ad_{x_1}, ad_{x_2}]$ is an inner derivation of the nilradical. Since the nilradical is abelian, we have

$$[A^1, A^2] = 0. \tag{3}$$

Also, since $x_\alpha \notin \mathfrak{nil}(\mathfrak{g})$, then A^α cannot be nilpotent. In fact A^1 and A^2 are linearly nilindependent and commute pairwise.

We perform a combination of changes of basis until we reach our desired form. For $i = 1,\dots,5$, and $\alpha = 1,2$, the following changes of basis preserve the nilradical:

(i) Absorbtion-type change of basis

$$\bar{x}_\alpha = x_\alpha + r^i_\alpha e_i \qquad r^i_\alpha \in \mathbb{R}.$$

(ii) A change of basis in X

$$\bar{x}_\alpha = G^\beta_\alpha x_\beta \qquad G \in GL(2,\mathbb{R}).$$

(iii) A change of basis in $\mathfrak{nil}(\mathfrak{g})$

$$\bar{e}_i = S^j_i e_j \qquad S \in GL(5,\mathbb{R}),$$

where $S = (S^j_i)$ is the automorphism that will change every A^α to a similar matrix $SA^\alpha S^{-1}$.

Thus our classification problem reduces to finding the derivations of the nilradical that are not nilpotent and that satisfy equation (3).

3 Classes of Solvable Algebras

We will determine all real solvable algebras $N = \mathbb{R}^5 \oplus X$ such that the $\dim X = 2$. The dimension of the center of $\mathfrak{g}$ is

$$\dim Z(\mathfrak{g}) \leq 2\dim \mathfrak{nil}(\mathfrak{g}) - \dim \mathfrak{g} = 3$$

(see Ref. [10]). The algebras that possess a center of dimension at least two are decomposable into a direct sum of lower-dimensional algebras [10]. Therefore, in the following, the classification problem is solved for the cases $\dim Z(\mathfrak{g}) = 0, 1$.

The derivation matrices A^α form an abelian subalgebra of $gl(5,\mathbb{R})$ and hence a subalgebra of some maximal abelian subalgebra. This maximal abelian subalgebra cannot be a maximal abelian nilpotent subalgebra; as a matter of fact it contains no nilpotent elements at all. In his Ph.D. dissertation, Ndogmo [14] (and later in [15]) outlines a technique to find the equivalent classes of nilindependent derivations $\{A^1, A^2\}$. What we mean by "equivalent classes" is

(i) The pair $\{y^1_1 A^1 + y^1_2 A2, y^2_1 A^1 + y^2_2 A2\}$, where $y^1_1 y^2_2 - y^1_2 y^2_1 \neq 0$, is equivalent to $\{A^1, A^2\}$.

(ii) The pair $\{SA^1S^{-1}, SA^2S^{-1}\}$, where S is the automorphism of the nilradical, is equivalent to $\{A^1, A^2\}$.

Using Ndogmo and Winternitz's notation, we list our A^1, A^2 in block-diagonal form by the dimension of each block. Namely, the

$$(u_1 u_2 \cdots u_i, v_1 v_2 \cdots v_j, w_1)$$

partition consists if i real lower triangular blocks of dimension $u_i \times u_i$, j real blocks of complex conjugate type of dimension $v_j \times v_j$ (v_j is even), and w_1 stands for the dimension of a one-dimensional zero block. Note that for our cases $w_1 = 0$ for the trivial center cases, and $w_1 = 1$ for the one-dimensional center case. In our case $\sum u_i + \sum v_j + w_1 = 5$.

3.1 Algebras with trivial center

For all of the block partitions in this section, we were able to find a change of basis that produces $[x_1, x_2] = 0$.

The Jacobi identity can give a nonlinear homogeneous system of equations on the free parameters. Each solution to that system will give a subcase for that partition. We list the matrices $A^\alpha = ad_{x_\alpha}|_{\mathfrak{nil}(\mathfrak{g})}$ for each case or subcase and give the conditions on those free parameters. We summarize our list of algebras in Table 1, suppressing the conditions.

(i) The $(11111, 0, 0)$ partition

For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 \\ 0 & 0 & 0 & b_4 & 0 \\ 0 & 0 & 0 & 0 & b_5 \end{pmatrix}.$$

To ensure a trivial center and an indecomposable algebra, we need $a_i^2 + b_i^2 \neq 0$ for $i = 3,4,5$. We denote this algebra by $N_{7,1}$.

(ii) The (1112,0,0) partition

For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & p_1 & a_4 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 \\ 0 & 0 & 0 & b_4 & 0 \\ 0 & 0 & 0 & p_2 & b_4 \end{pmatrix}.$$

To ensure a trivial center and an indecomposable algebra, we need $a_i^2 + b_i^2 \neq 0$ for $i = 3,4$. We denote this algebra $N_{7,2}$.

(iii) The (111,2,0) partition

For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & b_1 & c_1 \\ 0 & 0 & 0 & -c_1 & b_1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 \\ 0 & 0 & 0 & b_2 & c_2 \\ 0 & 0 & 0 & -c_2 & b_2 \end{pmatrix}.$$

To ensure a trivial center and an indecomposable algebra as well as a complex block, we need $a_3^2 + b_3^2 \neq 0$ and $c_1^2 + c_2^2 \neq 0$. We denote this algebra $N_{7,3}$.

(iv) The (122, 0, 0) partition

For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & p_2 & a_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & q_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & q_2 & b_3 \end{pmatrix}.$$

To ensure a trivial center and an indecomposable algebra, we need $a_3^2 + b_3^2 \neq 0$.

We denote this algebra $N_{7,4}$.

(v) The (12, 2, 0) partition

For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & q_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We denote this algebra $N_{7,5}$.

(vi) The (1, 22, 0) partition

For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & 0 & b_1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

We denote this algebra $N_{7,6}$.

(vii) The (113,0,0) partition

For this partition, the homogeneous nonlinear system imposed by the Jacobi identity has three independent solutions. Each solution will give us a subcase below. For all the subcases, we need $a_3^2 + b_3^2 \neq 0$ to ensure a trivial center and an indecomposable algebra.

(a) The first solution requires $p_1 = q_1 = 0$. For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 \\ 0 & 0 & p_2 & p_3 & a_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 \\ 0 & 0 & 0 & b_3 & 0 \\ 0 & 0 & q_2 & q_3 & b_3 \end{pmatrix}.$$

We denote this algebra by $N_{7,7}$.

(b) The second solution requires $p_1 = p_3 = 0$. For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 \\ 0 & 0 & p_2 & 0 & a_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 \\ 0 & 0 & q_1 & b_3 & 0 \\ 0 & 0 & q_2 & q_3 & b_3 \end{pmatrix}.$$

We denote this algebra by $N_{7,8}$.

(c) The third solution requires $q_3 = p_3 q_1$ and $p_1 = 1$. For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 1 & a_3 & 0 \\ 0 & 0 & p_2 & p_3 & a_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 \\ 0 & 0 & q_1 & b_3 & 0 \\ 0 & 0 & q_2 & q_3 & b_3 \end{pmatrix}.$$

We denote this algebra by $N_{7,9}$.

(viii) The (14,0,0) partition

For this partition, the homogeneous nonlinear system imposed by the Jacobi identity has nine independent solutions. Each solution will give us a case below.

(a) The first solution requires $q_4 q_6 \neq 0$, $p_1 = \frac{q_1 p_4}{q_4}$, $p_2 = -\frac{p_4 q_1 q_5 - p_4 q_2 q_6 - p_5 q_1 q_4}{q_4 q_6}$, and $p_6 = \frac{p_4 q_6}{q_4}$. For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 & 0 \\ 0 & p_2 & p_4 & 0 & 0 \\ 0 & p_3 & p_5 & p_6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & q_1 & 1 & 0 & 0 \\ 0 & q_2 & q_4 & 1 & 0 \\ 0 & q_3 & q_5 & q_6 & 1 \end{pmatrix}.$$

We denote this algebra by $N_{7,10}$.

(b) The second solution requires $p_1 = p_6 = q_1 = q_6 = 0$. For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & p_2 & p_4 & 0 & 0 \\ 0 & p_3 & p_5 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & q_2 & q_4 & 1 & 0 \\ 0 & q_3 & q_5 & 0 & 1 \end{pmatrix}.$$

We denote this algebra by $N_{7,11}$.

(c) The third solution requires $q_1q_4 \neq 0$, $p_6 = q_6 = 0$, $p_1 = \frac{q_1 p_4}{q_4}$, and $p_5 = \frac{p_4 q_5}{q_4}$.
For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 & 0 \\ 0 & p_2 & p_4 & 0 & 0 \\ 0 & p_3 & p_5 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & q_1 & 1 & 0 & 0 \\ 0 & q_2 & q_4 & 1 & 0 \\ 0 & q_3 & q_5 & 0 & 1 \end{pmatrix}.$$

We denote this algebra by $N_{7,12}$.

(d) The fourth solution requires $q_1q_5 \neq 0$, $p_4 = q_4 = 0$, and $p_1 = -\frac{p_2 q_6 - p_5 q_1 - p_6 q_2}{q_5}$.
For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 & 0 \\ 0 & p_3 & p_5 & p_6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & q_1 & 1 & 0 & 0 \\ 0 & q_2 & 0 & 1 & 0 \\ 0 & q_3 & q_5 & q_6 & 1 \end{pmatrix}.$$

We denote this algebra by $N_{7,13}$.

(e) The fifth solution requires $q_6 \neq 0$, $p_4 = q_4 = q_5 = 0$, and $p_2 = \frac{p_5 q_1 + p_6 q_2}{q_6}$.
For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 & 0 \\ 0 & p_3 & p_5 & p_6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & q_1 & 1 & 0 & 0 \\ 0 & q_2 & 0 & 1 & 0 \\ 0 & q_3 & 0 & q_6 & 1 \end{pmatrix}.$$

We denote this algebra by $N_{7,14}$.

(f) The sixth solution requires $q_1 \neq 0$, $p_4 = q_4 = q_5 = q_6 = 0$, and $p_5 = -\frac{q_2 p_6}{q_1}$.
For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 & 0 \\ 0 & p_3 & p_5 & p_6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & q_1 & 1 & 0 & 0 \\ 0 & q_2 & 0 & 1 & 0 \\ 0 & q_3 & 0 & 0 & 1 \end{pmatrix}.$$

We denote this algebra by $N_{7,15}$.

(g) The seventh solution requires $p_6q_5 \neq 0$, $q_1 = q_4 = q_6 = 0$, and $p_1 = \frac{q_2 p_6}{q_5}$. For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 & 0 \\ 0 & p_2 & p_4 & 0 & 0 \\ 0 & p_3 & p_5 & p_6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & q_2 & 0 & 1 & 0 \\ 0 & q_3 & q_5 & 0 & 1 \end{pmatrix}.$$

We denote this algebra by $N_{7,16}$.

(h) The eighth solution requires $p_6 \neq 0$ and $q_1 = q_2 = q_4 = q_5 = q_6 = 0$. For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 & 0 \\ 0 & p_2 & p_4 & 0 & 0 \\ 0 & p_3 & p_5 & p_6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & q_3 & 0 & 0 & 1 \end{pmatrix}.$$

We denote this algebra by $N_{7,17}$.

(i) The ninth solution requires $p_1 \neq 0$ and $p_6 = q_1 = q_4 = q_5 = q_6 = 0$. For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 & 0 \\ 0 & p_2 & p_4 & 0 & 0 \\ 0 & p_3 & p_5 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & q_2 & 0 & 1 & 0 \\ 0 & q_3 & 0 & 0 & 1 \end{pmatrix}.$$

We denote this algebra by $N_{7,18}$.

(ix) The (23, 0, 0) partition

For this partition, the homogeneous nonlinear system imposed by the Jacobi identity has three independent solutions. Each solution will give us a case below.

(a) The first solution requires $p_2 = q_2 = 0$. For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ p_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_3 & p_4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ q_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & q_3 & q_4 & 1 \end{pmatrix}.$$

We denote this algebra by $N_{7,19}$.

(b) The second solution requires $p_2 = p_4 = 0$. For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ p_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_3 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ q_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & q_2 & 1 & 0 \\ 0 & 0 & q_3 & q_4 & 1 \end{pmatrix}.$$

We denote this algebra by $N_{7,20}$.

(c) The third solution requires $q_4 = q_2 p_4$ and $p_2 = 1$. For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ p_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & p_3 & p_4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ q_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & q_2 & 1 & 0 \\ 0 & 0 & q_3 & q_4 & 1 \end{pmatrix}.$$

We denote this algebra by $N_{7,21}$.

(x) The (3, 2, 0) partition

For this partition, the homogeneous nonlinear system imposed by the Jacobi identity has three independent solutions. Each solution will give us a case below.

(a) The first solution requires $p_1 = q_1 = 0$. For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ p_2 & p_3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ q_2 & q_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We denote this algebra by $N_{7,22}$.

(b) The second solution requires $p_1 = p_3 = 0$. For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ p_2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ q_1 & 0 & 0 & 0 & 0 \\ q_2 & q_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We denote this algebra by $N_{7,23}$.

(c) The third solution requires $q_3 = p_3 q_1$ and $p_1 = 1$. For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ p_2 & p_3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ q_1 & 0 & 0 & 0 & 0 \\ q_2 & q_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We denote this algebra by $N_{7,24}$.

3.2 Algebras with one-dimensional center

In this section, the center of the Lie algebra $Z(\mathfrak{g}) = \langle e_5 \rangle$. For all of the block partitions in this section, we were able to find a change of basis that produces $[x_1, x_2] = e_5$.

Similarly, the Jacobi identity can give a nonlinear homogeneous system of equations on the free parameters. Each solution to that system will give a subcase for that partition. We list the matrices $A^\alpha = ad_{x_\alpha}|_{\mathfrak{nil}(\mathfrak{g})}$ for each case or subcase and give the conditions on those free parameters. We summarize our list of algebras in Table 2, suppressing the conditions.

(i) The (1111, 0, 1) partition

For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 \\ 0 & 0 & 0 & b_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

To ensure a one-dimensional center and an indecomposable algebra, we need $a_i^2 + b_i^2 \neq 0$ for $i = 3,4$. We denote this algebra by $N_{7,25}$.

(ii) The (2, 2, 1) partition

For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ p_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ q_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We denote this algebra by $N_{7,26}$.

(iii) The (13, 0, 1) partition

For this partition, the homogeneous nonlinear system imposed by the Jacobi identity has three independent solutions. Each solution will give us a case below.

(a) The first solution requires $p_1 = q_1 = 0$. For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & p_2 & p_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & q_2 & q_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We denote this algebra by $N_{7,27}$.

(b) The second solution requires $p_1 = p_3 = 0$. For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & q_1 & 1 & 0 & 0 \\ 0 & q_2 & q_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We denote this algebra by $N_{7,28}$.

(c) The third solution requires $q_3 = p_3 q_1$ and $p_1 = 1$. For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & p_2 & p_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & q_1 & 1 & 0 & 0 \\ 0 & q_2 & q_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We denote this algebra by $N_{7,29}$.

(iv) The (22,0,1) partition

For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ p_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ q_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & q_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We denote this algebra by $N_{7,30}$.

(v) The (112,0,1) partition

For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & p_1 & a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 \\ 0 & 0 & q_1 & b_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

To ensure a one-dimensional center and an indecomposable algebra, we need $a_3^2 + b_3^2 \neq 0$. We denote this algebra by $N_{7,31}$.

(vi) The (11,2,1) partition

For this case our adjoint matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We denote this algebra by $N_{7,32}$.

(vii) The (0, 4, 1) partition

For this case our adjoint matrices are given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ p_1 & p_3 & 0 & 1 & 0 \\ p_2 & p_4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ q_1 & -q_2 & 1 & 0 & 0 \\ q_2 & q_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We denote this algebra by $N_{7,33}$.

(viii) The (0,22,1) partition

For this case our adjoint matrices are given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_2 & 0 & 0 \\ 0 & 0 & 0 & b_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We denote this algebra by $N_{7,34}$.

4 Conclusion

This completes the classification for the seven-dimensional solvable Lie algebras with five-dimensional abelian nilradical. Significant progress has been made on the remaining five-dimensional nilradicals and will be submitted separately.

5 Acknowledgements

This work was partially supported by National Science Foundation grant DMS-1659113, which funds a Research Experiences for Undergraduates program at Grand Valley State University.

1 Bracket relations for the real seven-dimensional solvable Lie algebras with five-dimensional Abelian nilradical and a trivial center

Table 1: Non-zero bracket relations for the real seven-dimensional solvable Lie algebras with five-dimensional Abelian nilradical and a trivial center. The elements $\{e_1,\dots,e_5\}$ form a basis for the nilradical and $\{x_1,x_2\}$ are the remaining basis elements.

		$[x_\alpha,e_1]$	$[x_\alpha,e_2]$	$[x_\alpha,e_3]$	$[x_\alpha,e_4]$	$[x_\alpha,e_5]$
2*$N_{7,1}$	$[x_1,e_i]$	e_1		a_3e_3	a_4e_4	a_5e_5
2-7	$[x_2,e_i]$		e_2	b_3e_3	b_4e_4	b_5e_5
2*$N_{7,2}$	$[x_1,e_i]$	e_1		a_3e_3	$a_4e_4+p_1e_5$	a_4e_5
2-7	$[x_2,e_i]$		e_2	b_3e_3	$b_4e_4+p_2e_5$	b_4e_5
2*$N_{7,3}$	$[x_1,e_i]$	e_1		a_3e_3	$b_1e_4-c_1e_5$	$c_1e_4+b_1e_5$
2-7	$[x_2,e_i]$		e_2	b_3e_3	$b_2e_4-c_2e_5$	$c_2e_4+b_2e_5$
2*$N_{7,4}$	$[x_1,e_i]$	e_1	p_1e_3		$a_3e_4+p_2e_5$	a_3e_5
2-7	$[x_2,e_i]$		$e_2+q_1e_3$	e_3	$b_3e_4+q_2e_5$	b_3e_5
2*$N_{7,5}$	$[x_1,e_i]$	e_1	p_1e_3		$-e_5$	e_4
2-7	$[x_2,e_i]$		$e_2+q_1e_3$	e_3	e_4	e_5
2*$N_{7,6}$	$[x_1,e_i]$	e_1	$-e_3$	e_2	b_1e_4	b_1e_5
2-7	$[x_2,e_i]$				$-e_5$	e_4
2*$N_{7,7}$	$[x_1,e_i]$	e_1		$a_3e_3+p_2e_5$	$a_3e_4+p_3e_5$	a_3e_5
2-7	$[x_2,e_i]$		e_2	$b_3e_3+q_2e_5$	$b_3e_4+q_3e_5$	b_3e_5
2*$N_{7,8}$	$[x_1,e_i]$	e_1		$a_3e_3+p_2e_5$	a_3e_4	a_3e_5
2-7	$[x_2,e_i]$		e_2	$b_3e_3+q_1e_4+q_2e_5$	$b_3e_4+q_3e_5$	b_3e_5
2*$N_{7,9}$	$[x_1,e_i]$	e_1		$a_3e_3+e_4+p_2e_5$	$a_3e_4+p_3e_5$	a_3e_5
2-7	$[x_2,e_i]$		e_2	$b_3e_3+q_1e_4+q_2e_5$	$b_3e_4+q_3e_5$	b_3e_5
2*$N_{7,10}$	$[x_1,e_i]$	e_1	$p_1e_3+p_2e_4+p_3e_5$	$p_4e_4+p_5e_5$	p_6e_5	
2-7	$[x_2,e_i]$		$e_2+q_1e_3+q_2e_4+q_3e_5$	$e_3+q_4e_4+q_5e_5$	$e_4+q_6e_5$	e_5
2*$N_{7,11}$	$[x_1,e_i]$	e_1	$p_2e_4+p_3e_5$	$p_4e_4+p_5e_5$		
2-7	$[x_2,e_i]$		$e_2+q_2e_4+q_3e_5$	$e_3+q_4e_4+q_5e_5$	e_4	e_5
2*$N_{7,12}$	$[x_1,e_i]$	e_1	$p_1e_3+p_2e_4+p_3e_5$	$p_4e_4+p_5e_5$		
2-7	$[x_2,e_i]$		$e_2+q_1e_3+q_2e_4+q_3e_5$	$e_3+q_4e_4+q_5e_5$	e_4	e_5
2*$N_{7,13}$	$[x_1,e_i]$	e_1	$p_1e_3+p_2e_4+p_3e_5$	p_5e_5	p_6e_5	
2-7	$[x_2,e_i]$		$e_2+q_1e_3+q_2e_4+q_3e_5$	$e_3+q_5e_5$	$e_4+q_6e_5$	e_5
2*$N_{7,14}$	$[x_1,e_i]$	e_1	$p_1e_3+p_2e_4+p_3e_5$	p_5e_5	p_6e_5	
2-7	$[x_2,e_i]$		$e_2+q_1e_3+q_2e_4+q_3e_5$	e_3	$e_4+q_6e_5$	e_5
2*$N_{7,15}$	$[x_1,e_i]$	e_1	$p_1e_3+p_2e_4+p_3e_5$	p_4e_5	p_6e_5	
2-7	$[x_2,e_i]$		$e_2+q_1e_3+q_2e_4+q_3e_5$	e_3	e_4	e_5
2*$N_{7,16}$	$[x_1,e_i]$	e_1	$p_1e_3+p_2e_4+p_3e_5$	$p_4e_4+p_5e_5$	p_6e_5	
2-7	$[x_2,e_i]$		$e_2+q_2e_4+q_3e_5$	$e_3+q_5e_5$	e_4	e_5

Continued on next page

Table 1 – *Continued from previous page*

		$[x_\alpha, e_1]$	$[x_\alpha, e_2]$	$[x_\alpha, e_3]$	$[x_\alpha, e_4]$	$[x_\alpha, e_5]$
2*$N_{7,17}$	$[x_1, e_i]$	e_1	$p_1e_3 + p_2e_4 + p_3e_5$	$p_4e_4 + p_5e_5$	p_6e_5	
2-7	$[x_2, e_i]$		$e_2 + q_3e_5$	e_3	e_4	e_5
2*$N_{7,18}$	$[x_1, e_i]$	e_1	$p_1e_3 + p_2e_4 + p_3e_5$	$p_4e_4 + p_5e_5$		
2-7	$[x_2, e_i]$		$e_2 + q_2e_4 + q_3e_5$	e_3	e_4	e_5
2*$N_{7,19}$	$[x_1, e_i]$	$e_1 + p_1e_2$	e_2	p_3e_5	p_4e_5	
2-7	$[x_2, e_i]$	q_1e_2		$e_3 + q_3e_5$	$e_4 + q_4e_5$	e_5
2*$N_{7,20}$	$[x_1, e_i]$	$e_1 + p_1e_2$	e_2	p_3e_5		
2-7	$[x_2, e_i]$	q_1e_2		$e_3 + q_2e_4 + q_3e_5$	$e_4 + q_4e_5$	e_5
2*$N_{7,21}$	$[x_1, e_i]$	$e_1 + p_1e_2$	e_2	$e_4 + p_3e_5$	p_4e_5	
2-7	$[x_2, e_i]$	q_1e_2		$e_3 + q_2e_4 + q_3e_5$	$e_4 + q_4e_5$	e_5
2*$N_{7,22}$	$[x_1, e_i]$	$e_1 + p_2e_3$	$e_2 + p_3e_3$	e_3	$-e_5$	e_4
2-7	$[x_2, e_i]$	q_2e_3	q_3e_3		e_4	e_5
2*$N_{7,23}$	$[x_1, e_i]$	$e_1 + p_2e_3$	e_2	e_3	$-e_5$	e_4
2-7	$[x_2, e_i]$	$q_1e_2 + q_2e_3$	q_3e_3		e_4	e_5
2*$N_{7,24}$	$[x_1, e_i]$	$e_1 + e_2 + p_2e_3$	$e_2 + p_3e_3$	e_3	$-e_5$	e_4
2-7	$[x_2, e_i]$	$q_1e_2 + q_2e_3$	q_3e_3		e_4	e_5

2 Bracket relations for the real seven-dimensional solvable Lie algebras with five-dimensional Abelian nilradical and a one-dimensional center

Table 2: Non-zero bracket relations for the real seven-dimensional solvable Lie algebras with five-dimensional Abelian nilradical and a one-dimensional center $Z(\mathfrak{g}) = \langle e_5 \rangle$. For all of the following algebras, $[x_1, x_2] = e_5$.

		$[x_\alpha, e_1]$	$[x_\alpha, e_2]$	$[x_\alpha, e_3]$	$[x_\alpha, e_4]$
2*$N_{7,25}$	$[x_1, e_i]$	e_1		a_3e_3	a_4e_4
2-6	$[x_2, e_i]$		e_2	b_3e_3	b_4e_4
2*$N_{7,26}$	$[x_1, e_i]$	$e_1 + p_1e_2$	e_2	$-e_4$	e_3
2-6	$[x_2, e_i]$	q_1e_2		e_3	e_4
2*$N_{7,27}$	$[x_1, e_i]$	e_1	p_2e_4	p_3e_4	
2-6	$[x_2, e_i]$		$e_2 + q_2e_4$	$e_3 + q_3e_4$	e_4
2*$N_{7,28}$	$[x_1, e_i]$	e_1	p_2e_4		
2-6	$[x_2, e_i]$		$e_2 + q_1e_3 + q_2e_4$	$e_3 + q_3e_4$	e_4
2*$N_{7,29}$	$[x_1, e_i]$	e_1	$e_3 + p_2e_4$	p_3e_4	
2-6	$[x_2, e_i]$		$e_2 + q_1e_3 + q_2e_4$	$e_3 + q_3e_4$	e_4
2*$N_{7,30}$	$[x_1, e_i]$	$e_1 + p_1e_2$	e_2	p_2e_4	
2-6	$[x_2, e_i]$	q_1e_2		$e_3 + q_2e_4$	e_4
2*$N_{7,31}$	$[x_1, e_i]$	e_1		$a_3e_3 + p_1e_4$	a_3e_4
2-6	$[x_2, e_i]$		e_2	$b_3e_3 + q_1e_4$	b_3e_4
2*$N_{7,32}$	$[x_1, e_i]$	e_1		$-e_4$	e_3
2-6	$[x_2, e_i]$		e_2	e_3	e_4

Continued on next page

Table 2 – *Continued from previous page*

		$[x_\alpha, e_1]$	$[x_\alpha, e_2]$	$[x_\alpha, e_3]$	$[x_\alpha, e_4]$
2*$N_{7,33}$	$[x_1, e_i]$	$-e_2 + p_1e_3 + p_2e_4$	$e_1 + p_3e_3 + p_4e_4$	$-e_4$	e_3
2-6	$[x_2, e_i]$	$e_1 + q_1e_3 + q_2e_4$	$e_2 - q_2e_3 + q_1e_4$	e_3	e_4
2*$N_{7,34}$	$[x_1, e_i]$	$-e_2$	e_1	b_2e_3	b_2e_4
2-6	$[x_2, e_i]$			$-e_4$	e_3

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