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The group of monomial matrices

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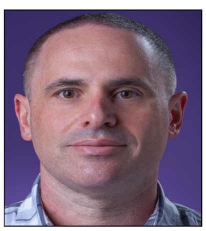

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Abstract

A recent result is used to give a brief proof of the well-known fact that the set of *monomial* matrices forms a subgroup of the set of invertible matrices. In addition, another proof is given of the result that the inverse of an invertible nonnegative matrix is nonnegative if and only if the matrix is monomial.

1 Introduction

In this note, we utilize a recent result [3, Lemma 3.3] to give a brief proof that the set of *monomial* matrices forms a subgroup of the set of invertible matrices. The result is well-known, but, to the best of our knowledge, a proof is not readily available in the literature and deserves wider circulation. In addition, we give an elementary proof that

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the inverse of an invertible nonnegative matrix is nonnegative if and only if the matrix is monomial.

2 Notation & Background

In this work, '$\mathbb{F}$' stands for $\mathbb{C}$ or $\mathbb{R}$. The algebra of n-by-n matrices with entries over $\mathbb{F}$ is denoted by $\mathsf{M}_n = \mathsf{M}_n(\mathbb{F})$ and the subset of invertible n-by-n matrices with entries from $\mathbb{F}$ is denoted by $\mathsf{GL}_n = \mathsf{GL}_n(\mathbb{F})$. The set of all n-by-1 column vectors is identified with the set of all ordered n-tuples with entries in $\mathbb{F}$ and thus denoted by $\mathbb{F}^n$. If $x \in \mathbb{F}^n$, then D_x denotes the diagonal matrix such that $d_{ii} = x_i$.

For $n \in \mathbb{N}$, denote by S_n the *symmetric group* of degree n. Given $\sigma \in S_n$, the *permutation matrix* with respect to σ, denoted by $P = P_\sigma \in \mathsf{M}_n$, is the n-by-n matrix such that $p_{ij} = \delta_{\sigma(i),j}$, where δ denotes the *Kronecker delta function*. The following facts concerning permutation matrices are well-known:

Proposition 1. *If σ, $\gamma \in S_n$, then:*

(i) $P_\sigma P_\gamma = P_{\gamma\circ\sigma}$;

(ii) $(P_\sigma)^{-1} = P_{\sigma^{-1}} = (P_\sigma)^\top$; *and*

(iii) *P is a permutation matrix if and only if P is a matrix with entries from $\{0,1\}$ and every row and every column of P contains exactly one nonzero entry.*

3 Monomial matrices

Definition 1. *If $A \in \mathsf{M}_n$, then A is called* monomial*, a* monomial matrix*, or a* generalized permutation matrix *if there is an invertible diagonal matrix D and a permutation matrix P such that $A = DP$. The set of all n-by-n monomial matrices is denoted by* $\mathsf{GP}_n = \mathsf{GP}_n(\mathbb{F})$

Remark 2. *If A is monomial with $A = DP$, then $a_{ij} = d_{ii}\delta_{\sigma(i),j}$. Following part (iii) of Proposition 1, A is monomial if and only if every row and every column of A contains exactly one nonzero entry.*

If $S \in \mathsf{GL}_n$, then the *relative gain array (RGA) of S*, denoted by $\Phi(S)$, is defined by $\Phi(S) = S \circ S^{-\top}$, where '$\circ$' denotes the *Hadamard* or entrywise product and $S^{-\top} := (S^{-1})^\top = (S^\top)^{-1}$. Johnson and Shapiro [4] showed that if $A = SD_xS^{-1}$, then

$$\Phi(S)x = \begin{bmatrix} a_{11} & \cdots & a_{nn} \end{bmatrix}^\top. \tag{1}$$

The following result, stated in slightly different terms, was established by Johnson and Paparella [3, Lemma 3.3] via the RGA.

Lemma 1. *If P is a permutation matrix and $x \in \mathbb{F}^n$, then $P^\top D_x P = D_y$, where $y := P^\top x$.*

Proof. Because a permutation similarity effects a simultaneous permutation of the rows and columns of a matrix, it follows that $P^\top D_x P$ is a diagonal matrix—say D_y.

Following (1) and part (ii) of Proposition 1,

$$y = \Phi(P^\top)x = [P^\top \circ (P^\top)^{-\top}]x = \left[P^\top \circ P^\top\right] x = P^\top x. \qquad \square$$

The following characterization is immediate from Lemma 1.

Corollary 1. *If $A \in \mathsf{M}_n$, then A is monomial if and only if $A = PD$, where D is an invertible diagonal matrix and P is a permutation matrix. Furthermore, if $A = D_x P$, where $x \in \mathbb{F}^n$, then $A = PD_y$, where $y := P^\top x$.*

Recall that if $A \in \mathsf{M}_n(\mathbb{R})$, then A is called *(entrywise) nonnegative* (respectively, *positive*), denoted by $A \geq 0$ (respectively, $A > 0$), if $a_{ij} \geq 0, 1 \leq i, j \leq n$ (respectively, $a_{ij} > 0, 1 \leq i, j \leq n$).

Lemma 2. *If A is monomial, then A is invertible and A^{-1} is monomial. Furthermore, if $A \geq 0$, then $A^{-1} \geq 0$.*

Proof. If A is monomial, then there is a vector $x \in \mathbb{F}^n$ with no zero entries and a permutation matrix P such that $A = D_x P$. By Corollary 1, $A = PD_y$, where $y = P^\top x$. The matrix A is invertible as it is the product of invertible matrices and

$$A^{-1} = (PD_y)^{-1} = (D_y)^{-1} P^{-1} = D_{y^{-1}} P^\top,$$

where $y^{-1} := \begin{bmatrix} x_1{}^{-1} & \cdots & x_n{}^{-1} \end{bmatrix}^\top$. By Definition 1, A^{-1} is a monomial matrix.

Notice that $A \geq 0$ if and only if $y > 0$. Thus, if $A \geq 0$, then $y^{-1} > 0$ and $A^{-1} \geq 0$ as it is the product of nonnegative matrices. $\square$

Theorem 3. *GP_n is a subgroup of GL_n.*

Proof. The identity matrix is clearly monomial, so GP_n is nonempty. In view of Lemma 2, it suffices to demonstrate closure. To this end, if $A, B \in \mathsf{GP}_n(\mathbb{F})$, then there are permutation matrices P and Q such that $A = D_x P$ and $B = D_y Q$. Thus,

$$\begin{aligned} AB &= (D_x P)(D_y Q) && \\ &= D_x((PD_y)Q) && \text{(associativity)} \\ &= D_x((D_z P)Q) && \text{(Lemma 1 with } z := Py) \\ &= (D_x D_z)(PQ) && \text{(associativity)} \\ &= D_{x \circ z}(PQ), && \end{aligned}$$

where '$\circ$' denotes the *Hadamard product*. $\square$

4 Nonnegative subgroups of Invertible Matrices

In 2013, Ding and Rhee [1] proved that an invertible matrix and its inverse are *stochastic* (i.e., entrywise nonnegative with rows summing to unity) if and only if the invertible matrix is a permutation matrix. In a subsequent work [2], they gave another proof of this result and used the result to show that an invertible matrix and its inverse are entrywise nonnegative if and only if the invertible matrix is monomial.

The import of the second result above can be gleaned from the following context. Recall that the set of invertible nonnegative matrices with matrix multiplication forms a *monoid*, i.e., it satisfies the *closure*, *associativity*, and *identity* group-axioms. However, as can be readily seen with two-by-two matrices, the inverse of an invertible nonnegative matrix need not be nonnegative.

The set of permutation matrices forms a nonnegative multiplicative subgroup of the set of invertible matrices, and it is natural to ask whether there are other nontrivial subsets of invertible nonnegative matrices that form a subgroup.

Theorem 3 above and Theorem 4 below provide the answer.

Theorem 4. *If A is nonnegative and invertible, then $A^{-1} \geq 0$ if and only if A is monomial.*

Proof. The sufficiency of this condition was shown in Lemma 2.

To demonstrate necessity, we modify the elementary argument given by Ding and Rhee [1] for stochastic matrices.

To this end, suppose that A is a nonnegative invertible matrix and that $A^{-1} \geq 0$. For convenience, write $B = A^{-1}$. Since $AB = I$, it follows that

$$\sum_{k=1}^{n} a_{ik} b_{kj} = \delta_{ij}.$$

In particular, if $i \neq j$, then

$$\sum_{k=1}^{n} a_{ik} b_{kj} = 0. \tag{2}$$

Fix $i \in \{1, \ldots, n\}$. Because A is invertible, the ith row of A must possess at least one positive entry—say a_{ir}. The nonnegativity of both matrices ensures that each summand on the left-hand side of (2) equals zero, i.e., $a_{ik} b_{kj} = 0$, $\forall k \in \{1, \ldots, n\}$. Since $a_{ir} > 0$, it follows that $b_{rj} = 0$ whenever $j \neq i$. Since the rth row of B cannot be zero, it must be the case that $b_{ri} > 0$.

Next, we show that a_{ir} is the only nonzero entry in the ith row of A. To the contrary, if $a_{is} > 0$, with $r \neq s$, then the argument above implies that $b_{sj} = 0$ whenever $j \neq i$ and $b_{si} > 0$. Thus, the rth and sth rows of B are (positive) multiples of each other, contradicting the invertibility of B.

Since $A^\top B^\top = I$, another application of the argument above with respect to the rth row of $A^\top$ demonstrates that a_{ir} is the only nonzero entry in the rth column of A. As i was arbitrary, the result applies to every row of A and because A is invertible, it must be the case that every row and every column of A contains exactly one nonzero entry, i.e., A is monomial. □

Corollary 2. *Any subgroup of invertible matrices in which every matrix is nonnegative must be a subgroup of the set of nonnegative momonial matrices.*

Bibliography

[1] J. Ding and N. H. Rhee. Teaching tip: when a matrix and its inverse are stochastic. *College Math. J.*, 44(2):108–109, 2013.

[2] J. Ding and N. H. Rhee. When a matrix and its inverse are nonnegative. *Missouri J. Math. Sci.*, 26(1):98–103, 2014.

[3] C. R. Johnson and P. Paparella. Perron spectratopes and the real nonnegative inverse eigenvalue problem. *Linear Algebra Appl.*, 493:281–300, 2016.

[4] C. R. Johnson and H. M. Shapiro. Mathematical aspects of the relative gain array $(A \circ A^{-T})$. *SIAM J. Algebraic Discrete Methods*, 7(4):627–644, 1986.