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Cram with Square Polyominoes

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Abstract

We will consider expansions of CRAM, a game frequently studied in the area of combinatorial game theory. The game of CRAM is classically played with dominoes, a type of polyomino. We will define CRAM WITH HIGHER POLYOMINOES and use efficient packing results to establish the outcome classes for several board shapes and choices of polyominoes.

1 Introduction and Background

Combinatorial Game theory studies finite strategy games of perfect information.

Definition 1.1 [from [1]]**.** *A* combinatorial game

(i) is finite*: the game must end and cannot end in a tie;*

(ii) is based on pure strategy*: it has no elements of chance such as coin tosses, dice rolls, or randomly drawn cards; nor of skill, such as darts or hockey;*

(iii) is sequential*: has alternating players who take turns moving;*

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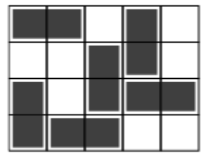

Figure 1: A game position of CRAM on a 5×4 board.

(iv) is a game of perfect information*: all players know all possible moves they can make as well as all moves the other players can make.*

Examples of combinatorial games include chess and go. Tic-tac-toe does not meet this definition since the game might end in a tie, while any games with secret information, such as most card games, are not included.

Our discussion focuses on games with only two players, a Left Player and a Right Player. We will assume the Left Player plays first, followed by the Right Player.

While analyzing these games, we assume both players *play perfectly*. In other words, if a player has a move or strategy that will allow the player to win, the player will play it.

A common example of a combinatorial game is CRAM. We define this game below, but it is well studied in, for example, [2] and [1].

Definition 1.2 *The game of* CRAM *is played on a board of square tiles. These boards can be of any size and arrangement. The two players alternately play a domino on two vertically or horizontally adjacent tiles. Any tile on the board can only hold a single cell of a domino. The game ends when the next player cannot play another domino. The last player to play wins.*

An example of a CRAM position can be seen in Figure 1. As each domino played is indistinguishable from all other dominoes and as both players have the same choice of possible moves, CRAM is an *impartial game*.

In Section 2 we introduce the idea of arbitrary polyominoes, and in Section 3 we extend the definition of CRAM to use these polyominoes instead of just dominoes. Then in Section 4 we study one version of this extended CRAM game using square polyominoes and are able to characterize the outcome of most games played on square boards. If the board is sufficiently small relative to the size of the piece then the first player to play will always have a winning strategy. If the board is square and exactly one less than three times the size of the piece then the second player has a winning strategy. Finally, if the board is square and three times the size of the piece or more, and the piece and board size are congruent mod 2 then the first player will have a winning strategy.

1.1 Outcome Classes

One final idea from combinatorial game theory is that of the outcome class of a game. The Fundamental Theorem of Combinatorial Games tells us how potential outcomes from these games are limited.

Theorem 1.3 (from [1]) *For any game played between two players, either the Left Player can force a win moving first or the Right Player can force a win moving second. Both cannot be true.*

This theorem implies that any game has one of four potential outcomes, Left Player will win regardless if she moves first or second, Right Player will win regardless if he moves first or second, the next player to play will win regardless of who that player is, or the second player to play will win. A second player win is also known as a previous player win since at any given moment in a game, the player who just moved is going to be the second player to play.

These outcomes are known as *outcome classes* and we will classify game positions according to their outcome classes. For a game position $\mathscr{G}$, the outcome class is denoted as $o(\mathscr{G})$. Because CRAM is an impartial game, any winning strategy for Left Player will work for Right Player, so our study focuses on the Next, $\mathscr{N}$, and Previous, $\mathscr{P}$, outcomes.

Definition 1.4 *A game $\mathscr{G}$ where the next player to play can force a win has the outcome class* Next*, denoted $o(\mathscr{G}) = \mathscr{N}$.*

A game $\mathscr{G}$ where the previous player who played can force a win has the outcome class Previous*, denoted $o(\mathscr{G}) = \mathscr{P}$.*

An example of a game with outcome class $\mathscr{N}$ is CRAM on a 1×4 board since the next player to play may place their domino on the middle two tiles. This ends the game as there are no more valid moves available.

An example of a game with the outcome class $\mathscr{P}$ is CRAM on a 2×2 board. Exactly two dominoes must be played before the game ends. As the first player cannot block the second player from playing a piece, the game has the outcome class $\mathscr{P}$.

2 Polyomino Preliminaries

Our goal is to extend CRAM to games using other pieces, so we take a moment to generalize the idea of a domino.

Definition 2.1. *A* polyomino, *P, is a finite collection of square cells such that each cell is vertically or horizontally adjacent to another cell.*

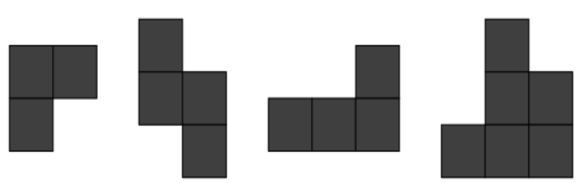

Figure 2: Examples of polyominoes.

See Figure 2 for examples of polyominoes. In this paper we will focus in particular on rectangular polyominoes.

Definition 2.2. *A* rectangular polyomino $R_{u,v}$ *is made of u columns of cells and v rows of cells.*

A square polyomino, *denoted* R_u*, is a rectangular polyomino with* $u = v$*.*

Examples are shown in Figure 3. We focus our attention specifically on square polyominoes, R_u.

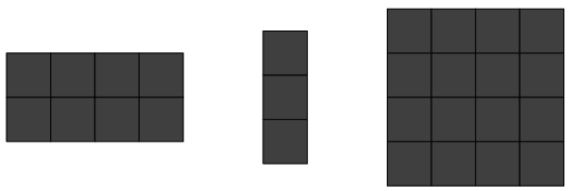

Figure 3: Examples of $R_{4,2}$, $R_{1,3}$, and R_4.

One common area of study for polyominoes is the study of packings and we will use some of these ideas in our results.

Definition 2.3 *A* packing *is an arrangement of polyominoes on a board such that no other polyominoes can be placed on the board.*

A packing number *is the number of polyominoes in a packing. An* efficient packing *on some board* $\mathscr{B}$ *of a polyomino P is a packing which uses the most possible copies of P. The number of copies used is the* efficient packing number *and is denoted* $\mathrm{p}_{\mathscr{B}}(P)$.

Similarly, a clumsy packing *on some board* $\mathscr{B}$ *of a polyomino P is a packing which uses the fewest possible copies of P. The number of copies used is the* clumsy packing number *and is denoted* $\mathrm{cp}_{\mathscr{B}}(P)$.

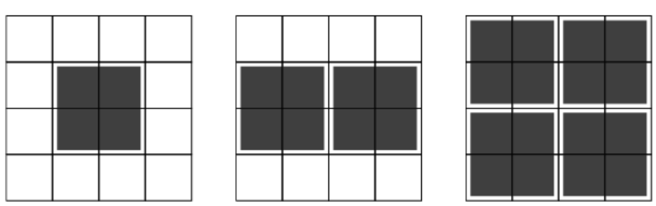

Figure 4: Examples of possible packings of a R_2 on a 4×4 board B. In this case, $\mathrm{cp}_{\mathscr{B}}(P)(R_2) = 1$ and $\mathrm{cp}_{\mathscr{B}}(P)(R_2) = 4$.

Additional resources that highlight packings include [5], [6], and [7]. Previous work on clumsy packings on finite boards specifically can be seen in [4]. An example of packings can be seen in Figure 4.

As an example, note that the efficient packing of $R_{u,v}$ on a rectangular board is trivial.

Observation 2.4. *For any $R_{u,v}$ efficient packing on a rectangular $n \times m$ board B, the* $\mathrm{p}_{\mathscr{B}}(R_{u,v}) = \lfloor \frac{n}{u} \rfloor \times \lfloor \frac{m}{v} \rfloor$. *In particular, if $n = m$ and $u = v$,* $\mathrm{p}_{\mathscr{B}}(R_u) = \lfloor \frac{n}{u} \rfloor^2$.

3 CRAM with Higher Polyominoes

We now extend the classical rules of CRAM to include general polyominoes and consider when we may determine the outcome class of such a game.

Definition 3.1. *For any polyomino P, we define the game of* CRAM WITH *P as played on a board B. Two players alternately play a free P on Bsuch that any tile of Bcan hold at most one cell of a P. The game ends when the next player cannot play another polyomino. The last player to play wins. In general, we call this class of games* CRAM WITH HIGHER POLYOMINOES.

CRAM WITH HIGHER POLYOMINOES was previously defined in [3] where the authors refer to the game as CRAMOMINOES.

We begin with some results relating game play in CRAM WITH HIGHER POLYOMINOES to packings of polyominoes. We then use these results to determine the outcome class of several games played on square boards.

3.1 CRAM and Packings

Since a game of CRAM WITH HIGHER POLYOMINOES ends when there is a packing, we will consider the packing of polyominoes on finite boards to determine the outcome of a game.

Lemma 3.2. *For a completed game position $\mathscr{G}$ of* CRAM WITH *P on a board $\mathscr{B}$, the number of moves to reach the position $\mathscr{G}$, $n(\mathscr{G})$, satisfies*

$$\mathrm{cp}_{\mathscr{B}}(P) \leq n(\mathscr{G}) \leq \mathrm{p}_{\mathscr{B}}(P).$$

Proof. As a game $\mathscr{G}$ of CRAM WITH *P* on a board Bmust end with a packing, the smallest possible number of moves is the clumsy packing number, $\mathrm{cp}_{\mathscr{B}}(P)$. The largest possible number of moves is the packing number, $\mathrm{p}_{\mathscr{B}}(P)$. Thus,

$$\mathrm{cp}_{\mathscr{B}}(P) \leq n(\mathscr{G}) \leq \mathrm{p}_{\mathscr{B}}(P).$$

□

We cannot simply use the clumsy packing number to determine the outcome of a game. For example, on a 9×2 board B, $\mathrm{cp}_{\mathscr{B}}(R_2) = 3$. However, if Left Player plays the move from Figure 5 the board must have a packing number of 4, which is greater than the clumsy packing number of the entire board.

Figure 5: An example of a move which forces the number of moves in the finished game to be greater than the clumsy packing number.

However, for CRAM WITH P on a board $\mathscr{B}$, if $\mathrm{cp}_{\mathscr{B}}(P) = \mathrm{p}_{\mathscr{B}}(P)$, then the number of moves in a completed game must satisfy $n(\mathscr{G}) = \mathrm{cp}_{\mathscr{B}}(P)$ by Lemma 3.2. In this case then, the outcome class of a game comes down to whether the packing numbers are odd or even.

Corollary 3.3. *For a game $\mathscr{G}$ of* CRAM WITH *P on a board $\mathscr{B}$, where $\mathrm{cp}_{\mathscr{B}}(P) = \mathrm{p}_{\mathscr{B}}(P)$, if $\mathrm{cp}_{\mathscr{B}}(P)$ is even, the game has the outcome class $\mathscr{P}$ and if $\mathrm{cp}_{\mathscr{B}}(P)$ is odd, the game has the outcome class $\mathscr{N}$.*

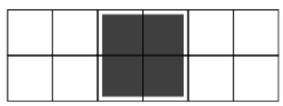

Figure 6: A game of CRAM with R_2 on a 6×2 board has the clumsy packing of 2, however, the game has the outcome class $\mathscr{N}$ since the first player may play as shown above. This leaves exactly two moves left and a win for the first player.

Unfortunately, if the $\mathrm{cp}_{\mathscr{B}}(P) \neq \mathrm{p}_{\mathscr{B}}(P)$ and if $\mathrm{cp}_{\mathscr{B}}(P)$ is odd, the game does not necessarily have the outcome class $\mathscr{N}$ as discussed in the previous example with Figure 5. Similarly, if the $\mathrm{cp}_{\mathscr{B}}(P) \neq \mathrm{p}_{\mathscr{B}}(P)$ and if $\mathrm{cp}_{\mathscr{B}}(P)$ is even, the game does not necessarily have the outcome class $\mathscr{P}$. An example of this can be seen in Figure 6.

The following observation is a trivial alternative for establishing specific cases of games with the outcome class $\mathscr{N}$ and $\mathscr{P}$. Typically, if one piece can be played to win the game, that game has the outcome class $\mathscr{N}$. The following observation provides a method for determining the outcome class of a game using the clumsy packing number to prove that one piece can win the game. This means we do not need to consider all other possible moves to determine the outcome. Similarly, if no pieces can be played to start the game, that game has the outcome class $\mathscr{P}$.

Observation 3.4. Let $\mathscr{G}$ be CRAM WITH P on a board $\mathscr{B}$, if $\mathrm{cp}_{\mathscr{B}}(P) = 1$ then, $o(\mathscr{G}) = \mathscr{N}$ and if $\mathrm{cp}_{\mathscr{B}}(P) = 0$ then, $o(\mathscr{G}) = \mathscr{P}$.

In the following section we will consider CRAM WITH P specifically when playing with square polyominoes.

4 $u \times u$ Square Cram

We consider CRAM WITH P using $u \times u$ square polyominoes, R_u.

In many of the following proofs, for a game of CRAM WITH R_u it will be useful to consider the center of a rectangular $a \times b$ board. For a square polyomino R_u, when u is odd, we will consider the single middlemost cell of the polyomino to be the center cell. When u is even, we will consider the middlemost 2×2 set of cells to be the center cells. An analogous idea used to define the center tile(s) on an $a \times b$ board when both a and b are either odd or even. If a is odd and b is even then the two tiles in the intersection of the center column with the two middlemost rows will be the center tiles. The case when b is odd and a is even is similar.

Definition 4.1. *We will say a player plays* centerish *if the number of center cells of a polyomino placed in the center tiles of the board is maximized.*

Note if both the board and polyomino have odd dimensions then there is only one play that would be considered centerish. Similarly, if both have even dimensions there is only one centerish play as all four center cells of the polyomino must be on all four center tiles of the board. When one has odd dimensions and the other has even dimensions there would be four plays that are considered centerish. See Figure 7 for an example.

In the following theorems, we establish the outcome for CRAM WITH R_u played on boards with sides less then or equal to $3u-1$ and square boards greater than $3u-1$ if the board and piece are both even or odd. This covers all games of CRAM WITH R_u played on square boards except when the size of the square board is even and u is odd or vice-versa.

Theorem 4.2. *Let $n, u \in \mathbb{Z}^+$, $u > 1$. If $u \leq n \leq 3u-2$, then* CRAM WITH R_u *on an $n \times n$ board has the outcome class $\mathscr{N}$.*

Proof. Assume $n, u \in \mathbb{Z}^+$. Let $\mathscr{G}$ be CRAM WITH R_u on a rectangular $n \times n$ board Bwhere $u \leq n \leq 3u-2$. If the Left Player plays centerish, the tiles without a cell create at most a $u-1$ gap between the played piece and one or two of the edges of Bas shown in Figure 7. As a $u-1$ gap cannot hold another polyomino, the $\mathrm{cp}_{\mathscr{B}}(R_n) = 1$. Therefore by Observation 3.4, $o(\mathscr{G}) = \mathscr{N}$. □

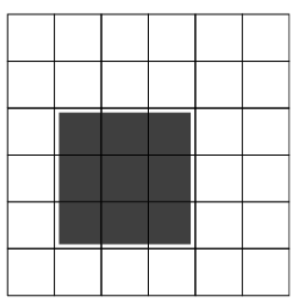

Figure 7: An example of CRAM WITH R_3 on a 6×6 board showing one of the ways for Left Player to play centerish.

Theorem 4.3. Let $n, u \in \mathbb{Z}^+$, $u > 1$. On an $n \times (3u-1)$ board, CRAM WITH R_u has the outcome class $\mathscr{P}$.

Proof. Let $\mathscr{G}$ be CRAM WITH R_u on a $n \times (3u-1)$ board B. If $n < u$ then no moves are possible and the game is a previous player win, so assume $n \geq u$. Let $\mathscr{C}$ be a $u \times (3u-1)$ portion of the board. Then $\text{cp}_{\mathscr{C}}(R_u) = \text{p}_{\mathscr{C}}(R_u) = 2$ and so any complete packing on $\mathscr{C}$ must contain 2 copies of R_u. As any set of u adjacent nonempty columns is analogous to $\mathscr{C}$, if the Left Player plays, the Right Player will always be able to play directly above or below that piece ending play in these columns. Therefore, for any move the Left Player makes, the Right Player has a response, and so $o(\mathscr{G}) = \mathscr{P}$. □

While Corollary 4.4 below follows directly from Theorem 4.2., we present an interesting alternate proof here.

Corollary 4.4. *Let $u \in \mathbb{Z}^+$, $u > 1$. Any game of* CRAM WITH R_u *on a* $(3u-1) \times (3u-1)$ *board has the outcome class* $\mathscr{P}$.

Proof. Let $\mathscr{G}$ be CRAM WITH R_u on a rectangular $(3u-1) \times (3u-1)$ board B.

Consider any packing of R_u on B. Note a $u \times u$ region at every corner of Bmust contain a cell of an R_u, an example is shown in Figure 8. If there is not a cell in this region, an R_u may be played in that region. Since no one R_u may overlap more than one of these regions, we need at least 4 copies of R_u to pack B. Thus, $\text{cp}_{\mathscr{B}}(R_u) \geq 4$.

By Observation 2.4, $\text{p}_{\mathscr{B}}(R_u) = \left\lfloor \frac{3u-1}{u} \right\rfloor^2 = 4$.

As $\text{cp}_{\mathscr{B}}(R_u) = \text{p}_{\mathscr{B}}(R_u) = 4$ and 4 is an even number, by Corollary 3.3, $o(\mathscr{G}) = \mathscr{P}$. □

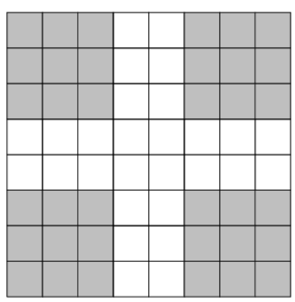

Figure 8: For a game of CRAM WITH R_3 on a 8×8 board, the shaded regions represent the 4 corner regions that must contain a piece of R_3.

In the following proofs we will discuss boards where the polyomino and length of the board are both even or both odd. In this situation, the piece can be played in the true center of the board so that the edge of the piece is the same number of cells away from the edge of the board on the top and bottom as well as the left and right. We will use this idea to discuss a mirroring strategy for playing the game.

Theorem 4.5. Let $u \in \mathbb{Z}^+$, $u > 1$, and let k be a non-negative, even integer. If $n = 3u + k$ then CRAM WITH R_u on an $n \times n$ board has the outcome class $\mathscr{N}$.

Proof. Given u, k as above, note u, n are both even or both odd.

Assume $u,n, \in \mathbb{Z}^+$. Let $n \geq 3u$ and u,n be even. Define $\mathscr{G}$ as CRAM WITH R_u on an $n \times n$ board B. As the size of Band R_u are both even, the Left Player may play in the true center of B. Then for any move Right Player makes, the Left Player has a response using rotational symmetry as follows: if a piece is played in one location, rotate the board 180° and place the polyomino in the same location. Thus after playing the first polyomino, the Left Player has a response to every Right Player move resulting in $o(\mathscr{G}) = \mathscr{N}$.

A similar argument holds when u,n are both odd. □

If the dimensions of the piece and the board are not both odd or even, then the first piece can sometimes be played centerish followed by a rotational mirroring strategy similar to that discussed in the proof of Theorem 4.6.

Theorem 4.6 Given $u,n,a \in \mathbb{Z}^+$. If u and n are both even or both odd with $u \leq n$ and $u \leq a \leq 3u-2$ then CRAM WITH R_u on a $n \times a$ board has the outcome class $\mathscr{N}$.

Proof. Assume $u,n,a \in \mathbb{Z}^+$. Let u and n be both even or both odd, $u \leq n$ and let $u \leq a \leq 3u-2$. Let $\mathscr{G}$ be CRAM WITH R_u on a $n \times a$ board. If the Left Player plays centerish, she splits the board into two disjoint $\frac{n-u}{2} \times a$ boards. As these are identical boards, through a mirroring strategy we see that $o(\mathscr{G}) = \mathscr{N}$. □

5 Conclusion and Future Work

Moving forward, there are a number other board sizes (including boards which are not rectangular) and different types of polyominoes to consider in the study of CRAM WITH HIGHER POLYOMINOES. In addition, expansions that put restrictions on the packings of the boards such as the ability for the polyomino/boards to rotate, adding a component of gravity, changing the type of board (such as a torus or Klein Bottle), and many more parameters could easily motivate future work in this area.

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