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Irrationality of the Riemann-Zeta function at the positive integers

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Yoochan Noh is a high school student at Korea International School, with a serious focus on college-level mathematics subjects. After graduation, he plans to pursue degrees in Applied Mathematics and Computer Science, driven by his passion for problem-solving and the transformative potential of technology. Inspired by previous small individual research projects, Yoochan sought to undertake a more substantial and profound research endeavor in mathematics. This led him to explore the complex and deep theme of the Riemann-Zeta function. Yoochan's aspiration extends beyond this research, as he hopes to engage in various types of research in the future, further expanding his understanding and contributing to the advancement of mathematical knowledge.

Abstract

The Riemann Zeta function, usually denoted by the Greek letter ζ, was defined in 1737 by a Swiss mathematician Leonhard Euler. This function is an infinite converging sum of powers of natural numbers, and it has explicit expressions in terms of π at positive even integers. In this paper we will discuss various irrationality proofs, focusing on irrationality of certain values of the Zeta function.

1 Introduction

We start with the definition of the Riemann-Zeta function (that we will just call Zeta function from now on).

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Definition 1.1. *On the complex half-plane* $\{(z) > 1 \mid z \in \mathbb{C}\}$ *the Riemann-Zeta function is defined by the following expression:*

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \ldots + \frac{1}{n^z} + \ldots$$

It is easy to show that the sum converges in this region.

Definition 1.1 will suffice for our purposes, but $\zeta(s)$ can be extended to the whole complex plane by [2]. Leonhard Euler did some basic computations with $\zeta(s)$. In particular, he famously solved the Basel Problem which is the question of determining the precise value of $\zeta(2)$. We cover his proof in the modern language in Section . Euler also generalized the computation to all positive even integers. One of the main results of this paper is a different proof of this formula that we give in Section 6.

The rest of the paper is organized as follows:

In Section 2 we discuss preliminaries needed to solve the Basel problem. In Section 3 we prove that certain radical expressions, and π, are irrational. In Section 4 we solve the Basel problem and compute $\zeta(4)$. In Section 5 we define Bernoulli numbers, an important preliminary for computing the Zeta function at the even integers. In Section 6 we discuss how the Zeta function at the even integers can be expressed in terms of Bernoulli numbers. In Section 7 we introduce the notion of being transcendental and explain that transcendentality of π [10] implies that $\zeta(2k)$, $k \geq 1$ is irrational. In Section 8 we prove that $\zeta(3)$ is irrational following [3, Theorem 2]. In Section 9 we show some advanced results, generalizations, and conjectures of the irrationality of the Zeta function.

2 Preliminaries

In order to prove irrationality of the Zeta function at certain values, it is necessary to understand certain preliminaries such as the infinite product formula for the sine function.

2.1 Logarithms of infinite products

Lemma 2.1. *For an infinite convergent product*

$$S = \prod_{n=1}^{\infty} a_n$$

it is always the case that

$$\log(S) = \sum_{n=1}^{\infty} \log(a_n)$$

Proof. If the infinite product converges to a positive number, continuity of the logarithm function permits the interchange of the limit and the logarithm. So,

$$\begin{aligned}\log \prod_{n=1}^{\infty} a_n &= \log \left(\lim_{k\to\infty} \prod_{n=1}^{k} a_n \right) \\ &= \lim_{k\to\infty} \log \left(\prod_{n=1}^{k} a_n \right) \\ &= \lim_{k\to\infty} \sum_{n=1}^{k} \log(a_n) \\ &= \sum_{n=1}^{\infty} \log(a_n)\end{aligned}$$

□

2.2 Derivatives of infinite sums

Lemma 2.2. *Suppose we have a sequence of functions f_n differentiable on $[a,b]$. If we have the series $\sum_{n=1}^{\infty} f_n(x)$ converging to $f(x)$ on $[a,b]$:*

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

and the series of derivatives $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on $[a,b]$, then we have

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) \qquad (a \le x \le b)$$

Proof. This is a standard result, see e.g. [11, Theorem 7.17]. □

2.3 Infinite product of the sine function

Theorem 2.3. *We have the equalities*

$$\sin(x) = x\left(\prod_{k=1}^{\infty}\left(1+\frac{x}{k\pi}\right)\left(1-\frac{x}{k\pi}\right)\right) = x\left(\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right)\right)$$

and

$$\frac{\sin(x)}{x} = \prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right)$$

that may be understood as expressing $\sin(x)$ as an infinite product over its roots at $n\pi$ for $n \in \mathbb{Z}$.

We refer to [8] for the proof.

3 Irrationality of radicals, and of π

In this section we give some elementary irrationality proofs. In particular, we prove that π is irrational using integral techniques. Somewhat similar, but much more advanced, methods will be used in Section 8 to prove that $\zeta(3)$ is irrational, a famous result due to Apéry [1].

Proposition 3.1. *For a prime number k, $\sqrt[n]{k}$ is an irrational number, for any $n \in \mathbb{Z}_{\geq 2}$.*

Proof. We first observe that $\sqrt[n]{k}$ is a root of the polynomial $x^n - k$. According to the rational root theorem, which uses ratios of the factors of the leading coefficient and the constant of a polynomial to determine its integer roots, a rational root of a polynomial with integer coefficients that is written in lowest terms $\frac{p}{q}$ must have denominator q that divides the leading coefficient, and numerator p that divides the constant coefficient. Here, $1 \equiv 0 \pmod q$, so q has to be 1, while $k \equiv 0 \pmod p$, so p is either k or 1. Therefore, any rational root of $x^n - k$ must be an integer.

The only way for $\sqrt[n]{k}$ to be an integer is if k is an n-th power of an integer, where $n \geq 2$. Since k is a prime number, it can only be expressed as p^1 when factorized. However, $1 \not\equiv 0 \ (\bmod n)$, so $\sqrt[n]{k}$ cannot be an integer, and hence not a rational number. □

Corollary 3.2. For prime numbers k and l, $\sqrt[n]{k + \sqrt[m]{l}}$ is an irrational number for any $n, m \in \mathbb{Z}_{>0}$

Proof. We can first assume that $\sqrt[n]{k + \sqrt[m]{l}}$ is rational, thereby stating that

$$\sqrt[n]{k + \sqrt[m]{l}} = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z}$$

This gives

$$\begin{aligned} k + \sqrt[m]{l} &= \frac{p^n}{q^n} \\ \sqrt[m]{l} &= \frac{p^n}{q^n} - k \end{aligned}$$

Since both $\frac{p^n}{q^n}$ and k are rational, it can be concluded that $\sqrt[m]{l}$ is also rational. However, have already shown that $\sqrt[m]{l}$ is irrational in Proposition 3.1, which contradicts the initial hypothesis. Therefore, $\sqrt[n]{k + \sqrt[m]{l}}$ has to be irrational. □

Theorem 3.3 (Irrationality of π)**.** *The number π is irrational.*

Proof. For any integrable function $f(x)$ by integration by parts we have:

$$\int f(x) \sin x \, dx = -f(x) \cos x + f'(x) \sin x - \int f''(x) \sin x \, dx$$

By using the values of $\sin(0) = 0$, $\cos(0) = 1$, $\sin(\pi) = 0$, and $\cos(\pi) = -1$,

$$\int_0^{\pi} f(x) \sin x \, dx = f(\pi) + f(0) - \int_0^{\pi} f''(x) \sin x \, dx$$

If $f(x)$ is a polynomial of degree $2n$, $n \in \mathbb{Z}_{>0}$, then repeating the calculation $n+1$ times would give

$$\int_0^{\pi} f(x) \sin x \, dx = F(\pi) + F(0) + \int_0^{\pi} f^{(2n+2)}(x) \sin x \, dx = F(\pi) + F(0) \qquad (1)$$

where $F(x) = f(x) - f''(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)$ and the last equality is from $f^{(2n+2)}(x) = 0$ (here $f^{(k)}(x)$ stands for the k-th derivative of $f(x)$).

Assume that π is rational, that is $\pi = \frac{p}{q}$ with $p, q \in \mathbb{Z}$ and $q \neq 0$. We will choose a particular polynomial $f(x)$ such that $F(0) + F(\pi)$ is an integer. Then, we will also show that $\int_0^\pi f(x) \sin x\, dx$ lies between 0 and 1, exclusively. Since no such integer can exist, this will obtain contradiction and π has to be irrational.
For $n \in \mathbb{Z}_{>0}$, let

$$f(x) = \frac{x^n (p - qx)^n}{n!}. \tag{2}$$

For $F(\pi) + F(0)$ to be an integer, we need to show that both $f^{(2n)}(\pi)$ and $f^{(2n)}(0)$ are integers.
For the chosen function $f(x)$,

$$f(\pi - x) = f\left(\frac{p}{q} - x\right) = \frac{\left(\frac{p}{q} - x\right)^n \left(p - q\left(\frac{p}{q} - x\right)\right)^n}{n!} = \frac{\left(\frac{p}{q} - x\right)^n (qx)^n}{n!} = \frac{x^n (p - qx)^n}{n!} = f(x)$$

Also, using the chain rule we see that for any $k \in \mathbb{Z}_{>0}$ we have

$$f^{(k)}(x) = (-1)^n f^{(k)}(\pi - x)$$

and

$$f^{(2n)}(0) = (-1)^{2n} f^{(2n)}(\pi) = f^{(2n)}(\pi)$$

So, if we show that $f^{(2n)}(0)$ is an integer, $f^{(2n)}(\pi)$ would also be an integer. We can express the function $f(x)$ in 2 ways:

$$f(x) = \frac{x^n (p - qx)^n}{n!} = \sum_{j=0}^{2n} \frac{c_j}{n!} x^j$$

for some $c_j \in \mathbb{Z}$. Also (according to the Taylor series),

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(2n)}(0)}{(2n)!} x^{2n}.$$

The coefficients at x^j for both equations should be equal.

$$\frac{c_{2n}}{n!} = \frac{f^{(2n)}(0)}{(2n)!}$$

Thus,

$$\frac{(2n)!}{n!} c_{2n} = f^{(2n)}(0)$$

Since $\frac{(2n)!}{n!} c_{2n}$ is an integer, then $f^{(2n)}(0)$ would also be an integer, which proves that $\int_0^\pi f(x) \sin x\, dx$ is an integer.
The next step is to show that (1) equates to a value strictly between 0 and 1.

$$f(x) = \frac{x^n (p - qx)^n}{n!} = \frac{x^n}{n!} (p - qx)^n$$

For $0 < x < \pi$, $\frac{x^n}{n!} > 0$, and $(p - qx)^n > 0$, so $f(x) > 0$ for $0 < x < \pi$. Also, since $\sin x > 0$ for $0 < x < \pi$ too, $f(x) \sin x > 0$ for the same domain, and therefore $\int_0^\pi f(x) \sin x\, dx > 0$. If the domain for x is $0 < x < \pi$, it can also be written as $0 < \pi - x < \pi$. By multiplying the 2 together, we get $0 < x(\pi - x) < \pi^2$.

Then,

$$0 < f(x) = \frac{x^n(p-qx)}{n!} = q^n \frac{x^n(\pi - x)^n}{n!} < q^n \frac{\pi^{2n}}{n!}$$

Since we are free to choose the value of n, we just need to show that

$$\lim_{n\to\infty} q^n \frac{\pi^{2n}}{n!} < \frac{1}{2} \tag{3}$$

Indeed, then we have $f(x) < \frac{1}{2}$ so

$$\int_0^\pi f(x) \sin x \, dx < \frac{1}{2} \int_0^\pi \sin x \, dx = 1.$$

To show (3), we just need to look at the Taylor series of the value of $e^{q\pi^2}$

$$e^{q\pi^2} = 1 + \frac{q\pi^2}{1!} + \frac{q^2\pi^4}{2!} + \frac{q^3\pi^6}{3!} + \dots$$

So, this infinite series converges to $e^{q\pi^2}$, a real number. However, if a particular infinite series $\sum_{n=0}^{\infty} a_n$ converges and $a_n > 0$, then

$$\lim_{n\to\infty} a_n = 0 < \frac{1}{2}$$

In our case,

$$\lim_{n\to\infty} q^n \frac{\pi^{2n}}{n!} = 0 < \frac{1}{2}$$

Therefore, using proof by contradiction, this shows that π is irrational. □

4 Elementary computation of $\zeta(2)$ and $\zeta(4)$.

A famous question, known as the Basel problem, is computing $\zeta(2)$. This result demonstrates that the infinite sum of the squares of the inverses of positive natural numbers is equal to the square of the number π divided by 6. We can write this as:

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} + \dots = \frac{\pi^2}{6}$$

The proof that we give below goes back to Euler [12, Theorem 1]. We also generalize the computation to calculate $\zeta(4)$.

4.1 Solving the Basel problem

Through the infinite product of the sine function formula seen in 2.3, we have concluded that:

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right) = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\left(1 - \frac{x^2}{4^2\pi^2}\right)\cdots \tag{4}$$

In addition to this, we can use the Taylor series to achieve the following equation:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \dots \tag{5}$$

Theorem 4.1 (Basel problem)**.** *We have*

$$\zeta(2) = \frac{\pi^2}{6} \tag{6}$$

Proof. The idea is to compare the coefficients at x^2 obtained using formulas (5) and (4). The coefficient at x^2 using (5) is $-\frac{1}{6}$. Let us compute the coefficient using (4). We have

$$\frac{\sin x}{x} = \lim_{n\to\infty} \left(\frac{\sin x}{x}\right)_n \tag{7}$$

where

$$\left(\frac{\sin x}{x}\right)_n := \prod_{k=1}^{n} \left(1 - \frac{x^2}{k^2\pi^2}\right) \tag{8}$$

We can investigate the finite order terms $\left(\frac{\sin x}{x}\right)_n$ starting with $n = 3$. Note that we have

$$\begin{aligned}
\left(\frac{\sin x}{x}\right)_3 &= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right) = \\
&= \left(1 - \frac{x^2}{\pi^2} - \frac{x^2}{2^2\pi^2} + T_4(2)x^4\right)\left(1 - \frac{x^2}{3^2\pi^2}\right) = 1 - \frac{x^2}{\pi^2} - \frac{x^2}{2^2\pi^2} - \frac{x^2}{3^2\pi^2} + T_4(3)x^4 + T_6(3)x^6 = \\
&= 1 - \frac{x^2}{\pi^2} - \frac{x^2}{2^2\pi^2} - \frac{x^2}{3^2\pi^2} + T(3)
\end{aligned}$$

where we denote by $T_m(n)$ the coefficient at x^m in the expansion of $\left(\frac{\sin x}{x}\right)_n$ and by $T(n)$ the sum $\sum_{m=4}^{2n} T_m(n)x^m$.

Similarly one can compute

$$\left(\frac{\sin x}{x}\right)_4 = 1 - \frac{x^2}{\pi^2} - \frac{x^2}{2^2\pi^2} - \frac{x^2}{3^2\pi^2} - \frac{x^2}{4^2\pi^2} + T(4)$$

$$\left(\frac{\sin x}{x}\right)_5 = 1 - \frac{x^2}{\pi^2} - \frac{x^2}{2^2\pi^2} - \frac{x^2}{3^2\pi^2} - \frac{x^2}{4^2\pi^2} - \frac{x^2}{5^2\pi^2} + T(5)$$

Using the same pattern, for an arbitrary $n \geq 3$ we get

$$\left(\frac{\sin x}{x}\right)_n = 1 - \frac{x^2}{\pi^2} - \frac{x^2}{2^2\pi^2} - \frac{x^2}{3^2\pi^2} - \dots - \frac{x^2}{n^2\pi^2} + T(n) \tag{9}$$

$$= T(n) + 1 - \left(\frac{1}{\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + \dots + \frac{1}{n^2\pi^2}\right)\left(x^2\right). \tag{10}$$

Now due to (7), by taking the limits of both sides, the coefficient at x^2 in (4) is equal to

$$-\left(\frac{1}{\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + \dots + \frac{1}{n^2\pi^2} + \cdots\right)$$

Therefore, we can equate the obtained coefficients to get

$$\frac{1}{\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + \dots + \frac{1}{n^2\pi^2} + \dots = \frac{1}{3!}.$$

Multiplying by π^2 both sides of the equation, we get:

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}+\cdots=\frac{\pi^2}{6}$$

which proves the result.

$\square$

4.2 Computing $\zeta(4)$

It has previously been stated that:

$$\zeta(2)=\sum_{i=1}^{\infty}\frac{1}{i^2} \tag{11}$$

Also, by following on with the definition of the Riemann Zeta function:

$$\zeta(4)=\sum_{i=1}^{\infty}\frac{1}{i^4} \tag{12}$$

Theorem 4.2. *We have*

$$\zeta(4)=\frac{\pi^2}{90} \tag{13}$$

Proof. Here, we are trying to modify equation (11) by squaring it, and also modify equation (7) so that it can be compared with the original equation and find the coefficient of x^4. Both of the modified equations will then be used to find $\zeta(4)$.

$$\begin{aligned}(\zeta(2))^2&=\left(\sum_{i=1}^{\infty}\frac{1}{i^2}\right)\left(\sum_{j=1}^{\infty}\frac{1}{j^2}\right)\\&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{i^2}\frac{1}{j^2}=\sum_{i=1}^{\infty}\frac{1}{i^4}+\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\frac{1}{i^2}\frac{1}{j^2}+\sum_{j=1}^{\infty}\sum_{i=j+1}^{\infty}\frac{1}{i^2}\frac{1}{j^2}\\&=\zeta(4)+2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\frac{1}{i^2}\frac{1}{j^2}\end{aligned}$$

Recall equation (8). From here, through the equation (and the proof of the Basel theorem above), we can see that the number of terms in the sum determining the coefficient at x^2 is $\binom{n}{1}$, and that the number of terms in the sum determining the coefficient at x^4 is $\binom{n}{2}$ For example,

$$\begin{aligned}\left(\frac{\sin x}{x}\right)_3&=1-\frac{x^2}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\right)+\frac{x^4}{\pi^4}\left(\frac{1}{1^2}\frac{1}{2^2}+\frac{1}{1^2}\frac{1}{3^2}+\frac{1}{2^2}\frac{1}{3^2}\right)-\cdots\\&=1-\frac{x^2}{\pi^2}\left(\sum_{i=1}^{3}\frac{1}{i^2}\right)+\frac{x^4}{\pi^4}\left(\sum_{1\le i<j\le 3}\frac{1}{i^2}\frac{1}{j^2}\right)-\cdots\end{aligned}$$

So, in general:

$$\left(\frac{\sin x}{x}\right)_n=1-\frac{x^2}{\pi^2}\left(\sum_{i=1}^{n}\frac{1}{i^2}\right)+\frac{x^4}{\pi^4}\left(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{i^2}\frac{1}{j^2}\right)-\cdots$$

In equation (7) we have established that:

$$\lim_{n\to\infty}\left(\frac{\sin x}{x}\right)_n = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - ...$$

Therefore,

$$\lim_{n\to\infty}\frac{1}{\pi^4}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{i^2}\frac{1}{j^2} = \frac{1}{5!}$$

and

$$\lim_{n\to\infty}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{i^2}\frac{1}{j^2} = \frac{\pi^4}{5!}. \tag{14}$$

It was previously shown that:

$$(\zeta(2))^2 = \zeta(4) + 2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\frac{1}{i^2}\frac{1}{j^2}$$

So, applying equation (14) we get

$$\zeta(4) = (\zeta(2))^2 - 2\frac{\pi^4}{5!}$$

But then

$$\zeta(4) = (\zeta(2))^2 - 2\frac{\pi^4}{5!} = \left(\frac{\pi^2}{6}\right)^2 - \frac{\pi^4}{60} = \frac{\pi^4}{36} - \frac{\pi^4}{60} = \frac{\pi^4}{90}$$

□

5 Bernoulli numbers

Definition 5.1. *Bernoulli numbers, often denoted B_n, are set of rational numbers that are often used in analysis. They are defined via the equation:*

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty}\frac{B_k}{k!}t^k$$

One can modify the equation

$$\left(\frac{e^t - 1}{t}\right)\left(\sum_{k=0}^{\infty}\frac{B_k}{k}t^k\right) = 1$$

by using the Taylor series expansion for $\frac{e^t-1}{t}$:

$$\left(\frac{1}{1!} + \frac{t}{2!} + \frac{t^2}{3!} + \dots\right)\left(\frac{B_0}{0!} + \frac{B_1}{1!}t + \dots\right) = 1$$

Here, we can see that $B_0 = 1$, and the coefficient of t^k becomes:

$$\frac{B_k}{k!}\frac{1}{1!} + \frac{B_{k-1}}{(k-1)!}\frac{1}{2!} + \frac{B_{k-2}}{(k-2)!}\frac{1}{3!} + \dots + \frac{B_0}{0!}\frac{1}{(k+1)!} = 0$$

Or,

$$B_0 = 1, \binom{k+1}{k}B_k + \binom{k+1}{k-1}B_{k-1} + \dots + \binom{k+1}{0}B_0 = 0$$

This gives a recursive way to compute B_k.

6 Computing $\zeta(2k)$ for $k \geq 1$

According to equation (4), we have:

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\left(1 - \frac{x^2}{4^2\pi^2}\right)\dots$$

Or, we can rewrite this equation by putting the natural log on both sides:

$$\log\left(\frac{\sin x}{x}\right) = \log\left(\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\left(1 - \frac{x^2}{4^2\pi^2}\right)\dots\right)$$

$$\log(\sin x) - \log x = \sum_{k=1}^{\infty} \log\left(1 - \frac{x^2}{k^2\pi^2}\right)$$

So, if we take the derivative on both sides of the equation in terms of x, we get (as long as $|x| < \pi$):

$$\cot x - \frac{1}{x} = \sum_{k=1}^{\infty}\left(-\frac{2x}{k^2\pi^2}\right)\frac{1}{1 - \frac{x^2}{k^2\pi^2}}$$

Therefore

$$\cot x = \frac{1}{x} + \sum_{k=1}^{\infty}\left(-\frac{2x}{k^2\pi^2}\right)\frac{1}{1 - \frac{x^2}{k^2\pi^2}} =$$

$$= \frac{1}{x} - 2\sum_{k=1}^{\infty}\left(-\frac{x}{k^2\pi^2}\right)\left(1 + \frac{x^2}{k^2\pi^2} + \frac{x^4}{k^4\pi^4} + \dots\right) =$$

$$= \frac{1}{x} - 2\left(\frac{\zeta(2)}{\pi^2}x + \frac{\zeta(4)}{\pi^4}x^3 + \frac{\zeta(6)}{\pi^6}x^5 + \dots\right)$$

and so

$$\frac{\cos x}{\sin x} = \frac{1}{x} - 2\sum_{k=1}^{\infty}\frac{\zeta(2k)}{\pi^{2k}}x^{2k-1} \tag{15}$$

According to Euler's formula, we have the equation:

$$e^{ix} = i\sin x + \cos x$$

If we substitute $-x$ instead of x into the equation, we get:

$$e^{-ix} = i\sin(-x) + \cos(-x) = -i\sin x + \cos x$$

Therefore,

$$\frac{e^{ix} + e^{-ix}}{2} = \cos x$$

And,

$$\frac{e^{ix} - e^{-ix}}{2i} = \sin x$$

So, using these two equations we have

$$\frac{\cos x}{\sin x} = \frac{\frac{e^{ix}+e^{-ix}}{2}}{\frac{e^{ix}-e^{-ix}}{2i}} = i\left(\frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}}\right) = i\left(\frac{e^{2ix} + 1}{e^{2ix} - 1}\right) = i\left(1 + \frac{2}{e^{2ix} - 1}\right) = i + \frac{1}{x}\left(\frac{2ix}{e^{2ix} - 1}\right)$$

Substituting into (15) gives

$$i+\frac{1}{x}\left(\frac{2ix}{e^{2ix}-1}\right)-\frac{1}{x}=-2\sum_{k=1}^{\infty}\frac{\zeta(2k)}{\pi^{2k}}x^{2k-1}$$

According to Definition 5.1

$$i+\frac{1}{x}\sum_{k=0}^{\infty}\frac{B_k}{k!}(2ix)^k-\frac{1}{x}=-2\sum_{k=1}^{\infty}\frac{\zeta(2k)}{\pi^{2k}}x^{2k-1}$$

This equation implies that

$$\sum_{k=2}^{\infty}\frac{B_k}{k!}(2ix)^k=-2\sum_{k=1}^{\infty}\frac{\zeta(2k)}{\pi^{2k}}x^{2k}$$

since the other coefficients in the sum on the left hand side cancel out. If we compare the coefficients of x^{2k} on both sides, we get

$$\frac{B_{2k}}{(2k)!}(2i)^{2k}=-2\frac{\zeta(2k)}{\pi^{2k}}$$

$$\zeta(2k)=\frac{B_{2k}}{(2k)!}(2i)^{2k}\frac{-\pi^{2k}}{2}$$

This simplifies to a general expression

$$\zeta(2k)=\frac{(-1)^{k+1}2^{2k-1}\pi^{2k}B_{2k}}{(2k)!} \tag{16}$$

generalizing (6) and (13).

7 Transcendentality of π and irrationality of $\zeta(2k),\ k\geq 1$

We now explain that (16) implies that the numbers $\zeta(2k),\ k\geq 1$ are irrational. Indeed, it is well-known that π is not just irrational but transcendental (refer to [10]).

Definition 7.1. *A number $\alpha\in\mathbb{R}$ is transcendental if it is not a root of any polynomial*

$$a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$$

with integer coefficients $a_i\in\mathbb{Z}$.

Proposition 7.2. *If $\alpha\in\mathbb{R}$ is transcendental, it is also irrational.*

Proof. Suppose that α is rational that is we have $\alpha=\frac{p}{q}$ for $p,q\in\mathbb{Z}$ with $q\neq 0$. Then α is a root of $qx-p=0$ contradicting Definition 7.1. □

According to (16), the numbers $\zeta(2k),\ k\geq 1$ are of the form $a\pi^{2k}$ for $a\in\mathbb{Q}$ (since $B_{2k}\in\mathbb{Q}$ for any $k\geq 1$). Therefore, irrationality of $\zeta(2k),\ k\geq 1$ follows from transcendentality of π and the following fact that generalizes Proposition 7.2.

Proposition 7.3. *If $\alpha\in\mathbb{R}$ is transcendental then α^k is irrational for any $k\geq 1$.*

Proof. The proof is similar to the proof of Proposition 7.2. Fix a $k\geq 1$ and suppose that α^k is rational that is we have $\alpha^k=\frac{p}{q}$ for $p,q\in\mathbb{Z}$ with $q\neq 0$. Then α is a root of qx^k-p contradicting Definition 7.1. □

8 Irrationality of $\zeta(3)$

In this section, we are going to integrate certain expressions involving logarithms and polynomials to show that $\zeta(3)$ is irrational. This is a famous result of Apéry [1]. We are going to follow the more elementary exposition of Beukers [3] while trying to give more details and some motivation.

First of all, we choose a certain polynomial and prove that it has integer coefficients. The choice of this polynomial is akin to the particular choice of the function $f(x)$ of (2) in the proof of Theorem 3.3 that π is irrational. We use the notation $f^{(n)}(x)$ for the n-th derivative of $f(x)$.

Lemma 8.1. *The polynomial $P_n(x) = \frac{1}{n!}(x^n(1-x)^n)^{(n)}$ has integer coefficients.*

Proof.

$$P_n(x) = \frac{1}{n!}\left(x^n(1-x)^n\right)^{(n)} = \frac{1}{n!}\left((x-x^2)^n\right)^{(n)}$$

According to the binomial theorem, this becomes

$$\frac{1}{n!}\left(\binom{n}{0}x^n(-x^2)^0 + \binom{n}{1}x^{n-1}(-x^2)^1 + \binom{n}{2}x^{n-2}(-x^2)^2 + \cdots + \binom{n}{n}x^0(-x^2)^n\right)^{(n)}$$

Let $a_i := (-1)^i\binom{n}{i}$, which is an integer. Then,

$$\begin{aligned} P_n(x) &= \left(\frac{a_0}{n!}x^n + \frac{a_1}{n!}x^{n+1} + \cdots + \frac{a_n}{n!}x^{2n}\right)^{(n)} \\ &= \frac{n!a_0}{n!0!} + \frac{(n+1)!a_1}{n!1!}x^1 + \cdots + \frac{(2n)!a_n}{n!n!}x^n \\ &= \binom{n}{0}a_0 + \binom{n+1}{1}a_1x^1 + \cdots + \binom{2n}{n}a_nx^n \end{aligned}$$

Since $\binom{p}{q} \in \mathbb{Z}$, $P_n(x)$ is an polynomial with integer coefficients.

□

Now, we prove that a certain double integral is a rational expression in terms of 1 and $\zeta(3)$, and provide a bound on the denominator of this expression. A more complicated integral of a similar form (with x^r and y^s replaced by $P_n(x)$ and $P_n(y)$) will be used later in the proof. This later integral may be regarded as the analog of $\int_0^\pi f(x)\sin x\,dx$ in the proof of Theorem 3.3 that π is irrational and the lemma below is analagous to proving that $\int_0^\pi f(x)\sin x\,dx$ is an integer.

Lemma 8.2. *Fix an $n \in \mathbb{Z}_{>0}$. Then for any $0 \le s \le r \le n$, we have*

$$\int_0^1\int_0^1 -\frac{\log(xy)}{1-xy}x^ry^s\,dx\,dy = \frac{A+B\cdot\zeta(3)}{(1,\ldots,n)^3}$$

for some $A, B \in \mathbb{Z}$

Proof. Consider the integral:

$$\int_0^1\int_0^1 -\frac{\log xy}{1-xy}x^ry^s\,dx\,dy$$

Note that the integral is proper since the integrand is bounded near $xy=1$ and convergent near $xy=0$ (even if $r=s=0$). Since for $|xy|<1$ we have

$$\frac{1}{1-xy} = 1+xy+x^2y^2+x^3y^3+\dots \tag{17}$$

the integral is equal to

$$-\int_0^1\int_0^1 \log(xy)x^ry^s(1+xy+x^2y^2+\dots)\,dx\,dy =$$

$$= -\int_0^1\int_0^1 \sum_{k=0}^{\infty} \log(xy)x^ry^sx^ky^k\,dx\,dy$$

$$= -\int_0^1 \sum_{k=0}^{\infty}\int_0^1 \log(xy)x^{r+k}y^{s+k}\,dx\,dy$$

(the interchange of the sum and the integral is justified since for any fixed $y\in(0,1)$ the convergence in (17) is uniform in $x\in[0,1]$). Integrating with respect to x and using integration by parts, it is easy to deduce:

$$-\int_0^1 \sum_{k=0}^{\infty}\int_0^1 \log(xy)x^{r+k}y^{s+k}\,dx\,dy = -\sum_{k=0}^{\infty}\left(\int_0^1 \frac{y^{s+k}\log(y)}{r+k+1} - \int_0^1 \frac{y^{s+k}}{(r+k+1)^2}dy\right)$$

By integrating with respect to y in a similar way, we get:

$$-\sum_{k=0}^{\infty}\left(\int_0^1 \frac{y^{s+k}\log(y)}{r+k+1} - \int_0^1 \frac{y^{s+k}}{(r+k+1)^2}dy\right) = \sum_{k=0}^{\infty}\left(\frac{1}{(r+k+1)(s+k+1)^2}+\frac{1}{(r+k+1)^2(s+k+1)}\right)$$

$$= \sum_{k=0}^{\infty}\left(\frac{r+s+2k+2}{(r+k+1)^2(s+k+1)^2}\right)$$

Since

$$\frac{1}{(s+k+1)^2}-\frac{1}{(r+k+1)^2} = \frac{(r-s)(r+s+2k+2)}{(r+k+1)^2(s+k+1)^2}$$

and

$$\sum_{k=0}^{\infty}\left(\frac{1}{(s+k+1)^2}-\frac{1}{(r+k+1)^2}\right) = \frac{1}{(s+1)^2}-\frac{1}{(r+1)^2}+\frac{1}{(s+2)^2}-\frac{1}{(r+2)^2}+\dots$$

We can conclude that for $r>s$:

$$\sum_{k=0}^{\infty}\left(\frac{r+s+2k+2}{(r+k+1)^2(s+k+1)^2}\right) = \frac{1}{r-s}\left(\sum_{k=0}^{\infty}\frac{1}{(s+k+1)^2}-\sum_{k=0}^{\infty}\frac{1}{(r+k+1)^2}\right) = \frac{1}{r-s}\sum_{k=1}^{r-s}\cdot\frac{1}{(s+k)^2}$$

where the last equality follows by cancelling out the terms of the two series. Also, since $r-s<n$, $(s+k)^2(r-s)$ would be a divisor of $(1,\dots,n)^3$ since $r<n$. Therefore,

$$\int_0^1\int_0^1 -\frac{\log xy}{1-xy}x^ry^s\,dx\,dy$$

is a rational number with denominator dividing $(1,\dots,n)^3$ for $r>s$.
On the other hand, for $r=s$, the equation becomes:

$$\sum_{k=0}^{\infty}\left(\frac{1}{(r+k+1)(s+k+1)^2}+\frac{1}{(r+k+1)^2(s+k+1)}\right) = 2\sum_{k=0}^{\infty}\frac{1}{(r+k+1)^3}$$

Since $\zeta(3)$ is equal to $\sum_{k=1}^{\infty} \frac{1}{k^3}$,

$$2\sum_{k=0}^{\infty} \frac{1}{(r+k+1)^3} = 2\left(\zeta(3) - \sum_{k=1}^{r} \frac{1}{k^3}\right) \tag{18}$$

This implies the result since every k^3 in the expression $\sum_{k=1}^{r} \frac{1}{k^3}$ divides $(1,\ldots,n)^3$. $\square$

To continue the analogy with the proof of Theorem 3.3 that π is irrational, we also need to be able to bound above the integral that we use for the irrationality proof. Recall that in that proof, we just wanted to show that $\int_0^{\pi} f(x)\sin x\, dx$ lies between 0 and 1; here the argument will be more complicated.

Lemma 8.3. *Fix an $n \in \mathbb{Z}_{>0}$. Let $P_n(x)$ be as in Lemma 8.1. Then we have*

$$\int_0^1\int_0^1 -\frac{\log(xy)}{1-xy} P_n(x)P_n(y)\, dx\, dy \le 2\left(\frac{1}{27}\right)^n \zeta(3)$$

Proof. Consider the integral:

$$\int_0^1\int_0^1 -\frac{\log(xy)}{1-xy} P_n(x)P_n(y)\, dx\, dy$$

Since

$$-\frac{\log(xy)}{1-xy} = \int_0^1 \frac{1}{1-(1-xy)z} dz \tag{19}$$

we can rewrite the integral as a triple integral:

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1-(1-xy)z}\, dz P_n(x)P_n(y)\, dx\, dy$$

We have

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1-(1-xy)z} P_n(x)P_n(y)\, dz\, dx\, dy = \frac{1}{n!}\int_0^1\int_0^1\int_0^1 \frac{1}{1-(1-xy)z} P_n(y)\, d\left((x^n(1-x)^n)^{(n-1)}\right) dy\, dz$$

Swapping the order of integration and integrating by parts with respect to x, we get

$$\frac{1}{n!}\int_0^1\int_0^1\int_0^1 yz\cdot\left(\frac{1}{1-(1-xy)z}\right)^2 (x^n(1-x)^n)^{(n-1)} P_n(y)\, dx\, dy\, dz$$

using the fact that $(x^n(1-x)^n)^{(n-1)}$ is 0 at $x=0$ and $x=1$. Integrating with respect to x by parts $n-1$ more times in a similar fashion, we get

$$\frac{1}{n!}\int_0^1\int_0^1\int_0^1 n!\frac{x^n y^n z^n (1-x)^n P_n(y)}{(1-(1-xy)z)^{n+1}}\, dx\, dy\, dz = \int_0^1\int_0^1\int_0^1 \frac{x^n y^n z^n (1-x)^n P_n(y)}{(1-(1-xy)z)^{n+1}}\, dx\, dy\, dz$$

We now make a change of variables $x=u,\ y=v,\ z=\frac{1-w}{1-(1-uv)w}$. One can check that this defines a differentiable bijective map $[0,1]^3 \to [0,1]^3$ with Jacobian

$$(u,v,w) = \frac{-uv}{(1-(1-uv)w)^2}$$

Let

$$f(x,y,z) = \frac{x^n y^n z^n (1-x)^n P_n(y)}{(1-(1-xy)z)^{n+1}}$$

By changing the variables in the integral (see [11]) we have (this requires a bit of computation)

$$\int_0^1\int_0^1\int_0^1 f(x,y,z)\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 f(x(u,v,w),y(u,v,w),z(u,v,w))|(u,v,w)|\,du\,dv\,dw =$$

$$= \int_0^1\int_0^1\int_0^1 (1-w)^n(1-u)^n \frac{P_n(v)}{1-(1-uv)w}\,du\,dv\,dw$$

Integrating with respect to v by parts n times (and switching the order of integration) similarly to before, this integral is equal to

$$\int_0^1\int_0^1\int_0^1 \frac{u^n(1-u)^n v^n(1-v)^n w^n(1-w)^n}{(1-(1-uv)w)^{n+1}}\,du\,dv\,dw$$

The integrand expression is easy to estimate. Indeed, we have

$$1-(1-uv)w = (1-w)+uvw \geq 2\sqrt{1-w}\sqrt{uvw}$$

on $[0,1]^3$ by arithmetic-geometric mean inequality. Therefore, we have

$$\frac{u(1-u)v(1-v)w(1-w)}{1-(1-uv)w} \leq \frac{1}{2}\sqrt{u}(1-u)\sqrt{v}(1-v)\sqrt{w(1-w)}$$

on $[0,1]^3$. The maximum of $g(t)=\sqrt{t}(1-t)$ for $t\in[0,1]$ occurs at $t=\frac{1}{3}$ and the maximum of $h(t)=t(1-t)$ for $t\in[0,1]$ occurs at $t=\frac{1}{2}$. This implies that

$$\frac{u(1-u)v(1-v)w(1-w)}{1-(1-uv)w} \leq \frac{1}{27}$$

Therefore, we have

$$\int_0^1\int_0^1\int_0^1 \frac{u^n(1-u)^n v^n(1-v)^n w^n(1-w)^n}{(1-(1-uv)w)^{n+1}}\,du\,dv\,dw =$$

$$\int_0^1\int_0^1\int_0^1 \left(\frac{u(1-u)v(1-v)w(1-w)}{1-(1-uv)w}\right)^n \frac{1}{1-(1-uv)w}\,du\,dv\,dw \leq$$

$$\left(\frac{1}{27}\right)^n \int_0^1\int_0^1\int_0^1 \frac{1}{1-(1-uv)w}\,du\,dv\,dw \leq$$

$$\left(\frac{1}{27}\right)^n \int_0^1\int_0^1 -\frac{\log(uv)}{(1-uv)}\,du\,dv = 2\left(\frac{1}{27}\right)^n \zeta(3)$$

where the penultimate inequality is by (19) and the last equality follows from (18) in the proof of Lemma 8.2. □

Finally, we are ready to show that $\zeta(3)$ is irrational.

Theorem 8.4. *$\zeta(3)$ is irrational.*

Proof. Consider the integral

$$I_n := \int_0^1\int_0^1 -\frac{\log(xy)}{1-xy} P_n(x)P_n(y)\,dx\,dy$$

from Lemma 8.3. Then there exist some $A', B' \in \mathbb{Z}$ such that

$$I_n = \frac{A' + B'\cdot\zeta(3)}{(1,\ldots,n)^3}$$

Indeed, if $P_n(x) = \sum_{i=0}^{n} b_i x^i$, then $b_i \in \mathbb{Z}$ by Lemma 8.1. But then we have

$$I_n = \sum_{r=0}^{n}\sum_{s=0}^{n} b_r b_s \int_0^1 \int_0^1 -\frac{\log(xy)}{1-xy} x^r y^s \, dx\, dy$$

by linearity and the claim follows from Lemma 8.2.

Suppose that $\zeta(3)$ is rational. Then we have $\zeta(3) = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ with $q > 0$. Consider $|A' + B'\zeta(3)|$. Note that all the terms in the integrand expression of I_n are positive on $(0,1)$ (one can check that $P_n(x)$ is a polynomial in $x(1-x)$ with positive coefficients) so $I_n \neq 0$. On one hand, we have

$$|A' + B'\zeta(3)| = |A' + B'\frac{p}{q}| = \frac{|A'q + B'p|}{q} \geq \frac{1}{q} \tag{20}$$

On the other hand,

$$|A' + B'\zeta(3)| = I_n(1,\dots,n)^3 \leq 2\left(\frac{1}{27}\right)^n \zeta(3)(1,\dots,n)^3 \tag{21}$$

It is enough to show that $\lim_{n\to\infty} 2\left(\frac{1}{27}\right)^n \zeta(3)(1,\dots,n)^3 = 0$. Indeed, then we have

$$2\left(\frac{1}{27}\right)^n \zeta(3)(1,\dots,n)^3 < \frac{1}{q}$$

for n large enough which is a contradiction with (20) and (21). However, it is well-known that the Prime Number Theorem [5] implies that $\lim_{n\to\infty} \sqrt[n]{(1,\dots,n)} = e$. But then

$$\lim_{n\to\infty} 2\left(\frac{1}{27}\right)^n \zeta(3)(1,\dots,n)^3 = \lim_{n\to\infty} 2\left(\frac{1}{27}\right)^n \zeta(3)e^{3n} = 2\left(\frac{e^3}{27}\right)^n \zeta(3) = 0$$

where the last equality is since $\frac{e^3}{27} < 1$. □

9 Advanced results

In this paper we have shown that the the Riemann-Zeta function is irrational at the even positive integers and gave an exposition of Beukers' proof [3] that $\zeta(3)$ is irrational. Not much is known about irrationality of the Riemann-Zeta function at odd integers $\zeta(2k+1)$, for $k > 1$. We finish this paper by listing some known results:

(i) Infinitely many of $\zeta(2k+1)$, for $k \geq 1$ are irrational, see [7].

(ii) It was shown in [13] that one of $\zeta(5), \zeta(7), \dots, \zeta(17), \zeta(19)$ is irrational. The same author also proved a stronger result that one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.

These partial result motivate the following conjecture, widely believed to be true but inaccessible with the current tools.

Conjecture 9.1. *All the $\zeta(2k+1), k \geq 1$ are irrational. Moreover, they are transcendental and algebraically independent from powers of π.*

Here being *algebraically independent* from powers of π means that no $\zeta(2k+1), k \geq 1$ is a root of any polynomial with coefficients of the form

$$a_0 + a_1\pi + a_2\pi^2 + \cdots + a_n\pi^n$$

for some $n \in \mathbb{N}$ and $a_i \in \mathbb{Z},\ 1 \leq i \leq n$.

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