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On the Cauchy Transform of the Complex Power Function

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Abstract

The integral $\int_{|z|=1} \frac{z^\beta}{z-\alpha} dz$ for $\beta = \frac{1}{2}$ has been comprehensively studied by Mortini and Rupp for pedagogical purposes. We write for a similar purpose, elaborating on their work with the more general consideration $\beta \in \mathbb{C}$. This culminates in an explicit solution

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in terms of the hypergeometric function for $|\alpha| \neq 1$ and any $\beta \in \mathbb{C}$. For rational β, the integral is reduced to a finite sum. A differential equation in α is derived for this integral, which we show has similar properties to the hypergeometric equation.

1 Introduction

The purpose of this paper is to investigate integrals of the form

$$\int_{|z|=1} \frac{z^\beta}{z-\alpha}\, dz. \tag{1}$$

Our personal interest in this type of integral stems from a recent paper due to Mortini and Rupp [1], in which the authors evaluate (1) for $\beta = \frac{1}{2}$ using various methods.

Initially we note that the function z^β must be defined, for general $\beta \in \mathbb{C}$, in terms of some branch of the complex logarithm. In our notation, for $0 < \theta < 2\pi$, $\log_\theta(z)$ will represent the branch of the complex logarithm with branch cut $\{re^{i\theta} : r \geq 0\}$; it is defined on the simply connected domain $\mathbb{C} \setminus \{re^{i\theta} : r \geq 0\}$, and we fix $\log_\theta(1) = 0$. Under these conditions our branch is

$$\log_\theta(z) = \ln|z| + i\arg_\theta(z)$$

where $\arg_\theta$ is the argument function with values in $(\theta - 2\pi, \theta)$.
This branch can be related to the branch of the square root discussed in [1] by taking $t_0 = \theta - 2\pi$.
We denote by $\mathrm{Arg}(z)$ the argument of z falling in the range $[0, 2\pi)$, and by $\arg(z)$ the equivalence class (modulo 2π) of all possible values for the argument of z. Any condition with $\arg(z)$ is considered satisfied if one representative satisfies the condition.

The implications of using a branch of the complex logarithm to define the complex power are that even when we choose $|\alpha| \neq 1$, the meromorphic function

$$m_{\alpha,\beta,\theta}(z) := \frac{z^\beta}{z-\alpha} = \frac{e^{\beta \log_\theta(z)}}{z-\alpha} \tag{2}$$

will not be analytic, or even *continuous*, on the boundary of the unit disk. This is due to the branch cut necessary for the $\log_\theta$ function used in (2). The discontinuity at the branch cut, although merely a jump, prevents a simple evaluation with direct application of Cauchy's Residue Theorem. Rather, one must proceed using different methods.

The main results of the paper are explicit expressions of (1) in the two cases of $|\alpha| > 1$ and $|\alpha| < 1$. Specifically, we prove:

Theorem 1. *When $|\alpha| > 1$,*

$$\int_{\partial\mathbb{D}} m_{\alpha,\beta,\theta} = \begin{cases} -2\pi i \alpha^\beta & \beta \in \mathbb{Z}_{<0}, \\ 0 & \beta = 0, \\ e^{i\beta\theta}\left(1 - e^{-2\pi i\beta}\right)\frac{1}{\beta}\left[1 - {}_2F_1(1,\beta;1+\beta;\alpha^{-1}e^{i\theta})\right] & \beta \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}. \end{cases}$$

When $|\alpha| < 1$,

$$\int_{\partial\mathbb{D}} m_{\alpha,\beta,\theta} = \begin{cases} 2\pi i \alpha^{\beta} & \beta \in \mathbb{Z}_{\geq 0}, \\ e^{i\beta\theta}\left(1 - e^{-2\pi i\beta}\right)\frac{1}{\beta}\,{}_2F_1(1,-\beta;1-\beta;\alpha e^{-i\theta}) & \beta \in \mathbb{C}\setminus\mathbb{Z}_{\geq 0}. \end{cases}$$

In §2, the unit circle is approximated with a contour of integration which avoids the branch cut in order to derive an equation involving (1). The connection between (1) and the hypergeometric function, ${}_2F_1$, is made in §3 through the identification of a core integral in §3.1. In §4, series manipulation leads to the proof of Theorem 1. The particular case when $\beta \in \mathbb{Q}\setminus\mathbb{Z}$ is further simplified in §5, and in §6 we include a derivation of a differential equation for which (1) is a solution. Provided in §7, the appendix, is a discussion of measure theory topics leading up to the statement of Lebesgue's Dominated Convergence Theorem; adequate references are cited there for the curious reader.

2 Contour Method

We first extend §1 in [1], evaluating (1) via contour integration. For this section alone (§2) it is additionally assumed that $\mathrm{Arg}(\alpha) \neq \theta$ and $\alpha \neq 0$, so that α does not lie on the branch cut. Furthermore, we assume that $\Re(\beta) > 0$, as this condition will be necessary for certain bounds. The purpose of this section is to prove the following lemma:

Lemma 1. *If* $\mathrm{Arg}(\alpha) \neq \theta$, *and* $\Re(\beta) > 0$, *then for* $0 < |\alpha| < 1$,

$$\int_{\partial\mathbb{D}} m_{\alpha,\beta,\theta} = 2\pi i\alpha^{\beta} + e^{i\beta\theta}(1 - e^{-2\pi i\beta})\int_0^1 \frac{e^{\beta\ln t}}{t - \alpha e^{-i\theta}}\,dt,$$

and for $|\alpha| > 1$,

$$\int_{\partial\mathbb{D}} m_{\alpha,\beta,\theta} = e^{i\beta\theta}(1 - e^{-2\pi i\beta})\int_0^1 \frac{e^{\beta\ln t}}{t - \alpha e^{-i\theta}}\,dt.$$

Proof. There are 3 main steps:

§2.1) constructing a proper contour;

§2.2) finding singularities and computing their residues;

§2.2) using limits to derive a useful equation.

The lemma follows from plugging (11), (35), (13), (38), and (22) all back into (9). □

2.1 Constructing the Contour

Take a branch of the complex logarithm $\log_{\theta}$ in the definition of z^{β}, and let the contour of integration $\Gamma_{\varepsilon,\theta,\rho}$ consist of:

a) the line segment $L_{\varepsilon,\theta,\rho} := \{z \in \mathbb{C} : \rho \leq |z| \leq 1, \arg z = \theta + \varepsilon\}$,

b) the arc $C_{\varepsilon,\theta} := \{z \in \mathbb{C} : |z| = 1, \theta + \varepsilon \leq \arg z \leq \theta + 2\pi - \varepsilon\}$,

c) the line segment $M_{\varepsilon,\theta,\rho} := \{z \in \mathbb{C} : 1 > |z| > \rho, \arg z = \theta + 2\pi - \varepsilon\}$,

d) the arc $D_{\varepsilon,\theta,\rho} := \{z \in \mathbb{C} : |z| = \rho, \theta + 2\pi - \varepsilon \geq \arg z \geq \theta + \varepsilon\}$, oriented as usual, with the bounded region enclosed on the left as we trace the contour. For example, for the principal branch of log ($\log_\pi$ in our notation), the contour is as in Figure 2.1. Under this definition, we have

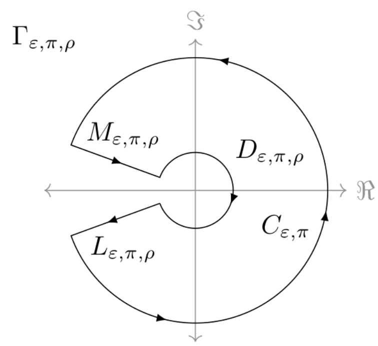

Figure 2.1: Contour for $\theta = \pi$.

$$\int_{\Gamma_{\varepsilon,\theta,\rho}} m_{\alpha,\beta,\theta} = \int_{C_{\varepsilon,\theta}} m_{\alpha,\beta,\theta} + \int_{D_{\varepsilon,\theta,\rho}} m_{\alpha,\beta,\theta} + \int_{L_{\varepsilon,\theta,\rho}} m_{\alpha,\beta,\theta} + \int_{M_{\varepsilon,\theta,\rho}} m_{\alpha,\beta,\theta}. \tag{3}$$

One can choose any parameterization of the four curves, noting that smooth equivalence of parameterizations will guarantee generality. In particular, we choose

a) $L_{\varepsilon,\theta,\rho}$: $z(t) = te^{i(\theta+\varepsilon)}$ for $\rho \le t \le 1$,
$$\int_{L_{\varepsilon,\theta,\rho}} m_{\alpha,\beta,\theta}(z)\,dz = \int_\rho^1 m_{\alpha,\beta,\theta}(te^{i(\theta+\varepsilon)})e^{i(\theta+\varepsilon)}\,dt; \tag{4}$$

b) $C_{\varepsilon,\theta}$: $z(t) = e^{it}$ for $\theta+\varepsilon \le t \le \theta+2\pi-\varepsilon$,
$$\int_{C_{\varepsilon,\theta}} m_{\alpha,\beta,\theta}(z)\,dz = \int_{\theta+\varepsilon}^{\theta+2\pi-\varepsilon} m_{\alpha,\beta,\theta}(e^{it})\; ie^{it}dt; \tag{5}$$

c) $M_{\varepsilon,\theta,\rho}$: $z(t) = te^{i(\theta+2\pi-\varepsilon)}$ for $1 \ge t \ge \rho$,
$$\int_{M_{\varepsilon,\theta,\rho}} m_{\alpha,\beta,\theta}(z)\,dz = \int_1^\rho m_{\alpha,\beta,\theta}(te^{i(\theta+2\pi-\varepsilon)})e^{i(\theta+2\pi-\varepsilon)}\,dt; \tag{6}$$

d) $D_{\varepsilon,\theta,\rho}$: $z(t) = \rho e^{it}$ for $\theta+2\pi-\varepsilon \ge t \ge \theta+\varepsilon$,
$$\int_{D_{\varepsilon,\theta,\rho}} m_{\alpha,\beta,\theta}(z)\,dz = \int_{\theta+2\pi-\varepsilon}^{\theta+\varepsilon} m_{\alpha,\beta,\theta}(\rho e^{it})\; i\rho e^{it}dt. \tag{7}$$

2.2 Applying the Residue Theorem

Applying Cauchy's Residue Theorem requires computing residues for singularities contained within the contour. To compute the residues of the meromorphic function $m_{\alpha,\beta,\theta}(z)$ defined in (2), first note that $e^{\beta \log_\theta(z)}$ is analytic in $\mathbb{C} \setminus \{re^{i\theta} \in \mathbb{C} : r \geq 0\}$, so the only singularity of $m_{\alpha,\beta,\theta}$ is at α, and this singularity only becomes relevant when $|\alpha| \leq 1$. This singularity is a simple pole, since

$$\lim_{z\to\alpha}(z-\alpha)m_{\alpha,\beta,\theta}(z) = \lim_{z\to\alpha} e^{\beta \log_\theta(z)} = \alpha^\beta \neq 0 \tag{8}$$

but

$$\lim_{z\to\alpha}(z-\alpha)^2 m_{\alpha,\beta,\theta}(z) = \lim_{z\to\alpha}(z-\alpha)e^{\beta \log_\theta(z)} = 0.$$

Evaluating as in (8), the residue at α is found to be α^β. In order to derive an equation involving (1), one might consider first taking the limit $\varepsilon \to 0^+$ and then $\rho \to 0^+$ in (3):

$$\lim_{\rho\to 0^+}\lim_{\varepsilon\to 0^+}\int_{\Gamma_{\varepsilon,\theta,\rho}} m_{\alpha,\beta,\theta} = \lim_{\rho\to 0^+}\lim_{\varepsilon\to 0^+}\left[\int_{C_{\varepsilon,\theta}} m_{\alpha,\beta,\theta} + \int_{D_{\varepsilon,\theta,\rho}} m_{\alpha,\beta,\theta} + \int_{L_{\varepsilon,\theta,\rho}} m_{\alpha,\beta,\theta} + \int_{M_{\varepsilon,\theta,\rho}} m_{\alpha,\beta,\theta}\right]. \tag{9}$$

Since the contour $\Gamma_{\varepsilon,\theta,\rho}$ in (9) lies in the interior of the simply connected domain of $\log_\theta$ whenever $\varepsilon, \rho > 0$, $m_{\alpha,\beta,\theta}$ is analytic on the path of integration so long as α does not lie on $\Gamma_{\varepsilon,\theta,\rho}$. In this case, Cauchy's Residue Theorem applies and so

$$\int_{\Gamma_{\varepsilon,\theta,\rho}} m_{\alpha,\beta,\theta} = 2\pi i\, n(\Gamma_{\varepsilon,\theta,\rho},\alpha)\,\mathrm{Res}(m_{\alpha,\beta,\theta},\alpha) = 2\pi i \alpha^\beta\; n(\Gamma_{\varepsilon,\theta,\rho},\alpha)$$

where $n(\Gamma_{\varepsilon,\theta,\rho},\alpha)$ is the winding number of $\Gamma_{\varepsilon,\theta,\rho}$ around α. Note that by definition of the contour, and because $\mathrm{Arg}(\alpha) \neq \theta$ by assumption, we have

$$n(\Gamma_{\varepsilon,\theta,\rho},\alpha) = \begin{cases} 1 & \text{if } 0 < \varepsilon < \min_{\arg(\alpha)}\{|\arg(\alpha)-\theta|\} \text{ and } 0 < \rho < |\alpha| < 1, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

where the notation $\min_{\arg(\alpha)}$ in (10) denotes that the minimum is taken over all possible representatives of $\arg(\alpha)$. It follows that

$$\lim_{\varepsilon\to 0^+} n(\Gamma_{\varepsilon,\theta,\rho},\alpha) = \begin{cases} 1 & \text{if } 0 < \rho < |\alpha| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

since ε can certainly be made smaller than $|\arg(\alpha)-\theta| > 0$, and that

$$\lim_{\rho\to 0^+}\lim_{\varepsilon\to 0^+} n(\Gamma_{\varepsilon,\theta,\rho},\alpha) = \begin{cases} 1 & \text{if } 0 < |\alpha| < 1, \\ 0 & \text{otherwise} \end{cases}$$

since ρ can certainly be made smaller than $|\alpha| > 0$. Therefore

$$\lim_{\rho\to 0^+}\lim_{\varepsilon\to 0^+}\int_{\Gamma_{\varepsilon,\theta,\rho}} m_{\alpha,\beta,\theta} = \begin{cases} 2\pi i \alpha^\beta & \text{if } 0 < |\alpha| < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

In evaluating $\lim_{\rho\to 0^+}\lim_{\varepsilon\to 0^+}\int_{C_{\varepsilon,\theta}} m_{\alpha,\beta,\theta}$, we use (5) to express

$$\begin{aligned}
\lim_{\rho\to 0^+}\lim_{\varepsilon\to 0^+}\int_{C_{\varepsilon,\theta}} m_{\alpha,\beta,\theta} &= \lim_{\rho\to 0^+}\lim_{\varepsilon\to 0^+}\int_{\theta+\varepsilon}^{\theta+2\pi-\varepsilon} m_{\alpha,\beta,\theta}(e^{it})\, ie^{it}\, dt,\\
&= \lim_{\varepsilon\to 0^+}\int_{\theta+\varepsilon}^{\theta+2\pi-\varepsilon} m_{\alpha,\beta,\theta}(e^{it})\, ie^{it}\, dt,\\
&= \mathscr{P}\mathscr{V}\int_{\theta}^{\theta+2\pi} m_{\alpha,\beta,\theta}(e^{it})\, ie^{it}\, dt,\\
&= \int_{\theta}^{\theta+2\pi} m_{\alpha,\beta,\theta}(e^{it})\, ie^{it}\, dt, \qquad (12)\\
&= \int_{|z|=1} m_{\alpha,\beta,\theta}(z)\, dz. \qquad (13)
\end{aligned}$$

To see that the value of the improper integral in (12) is the same as its principal value, note that whenever an improper integral converges, its principal value converges as well (and to the same value). By convention, (12) is evaluated as

$$\int_{\theta}^{\theta+2\pi} m_{\alpha,\beta,\theta}(e^{it})\, ie^{it}\, dt = \lim_{\varepsilon\to 0}\int_{\theta+\varepsilon}^{\theta+\pi} m_{\alpha,\beta,\theta}(e^{it})\, ie^{it}\, dt + \lim_{\varepsilon'\to 0}\int_{\theta+\pi}^{\theta+2\pi-\varepsilon'} m_{\alpha,\beta,\theta}(e^{it})\, ie^{it}\, dt. \qquad (14)$$

It suffices to show that $g(t) = m_{\alpha,\beta,\theta}(e^{it})\, ie^{it}$ is bounded on $[\theta,\theta+2\pi]$ in order for the right hand side of (14) to converge, and thus for the desired improper integral to converge. We first bound the real part of $\beta\log_\theta(z)$, noting that

$$\begin{aligned}
\beta\log_\theta(z) &= [\Re(\beta)+i\Im(\beta)]\,[\ln|z|+i\arg_\theta(z)]\,,\\
&= [\Re(\beta)\ln|z| - \Im(\beta)\arg_\theta(z)] + i[\Re(\beta)\arg_\theta(z)+\Im(\beta)\ln|z|], \qquad (15)
\end{aligned}$$

where $\arg_\theta := \Im(\log_\theta)$. Since we fix $\log_\theta(1) = 0$ for every $0<\theta<2\pi$, the continuity of $\log_\theta$ on its simply connected domain implies that

$$-2\pi < \arg_\theta(z) < 2\pi$$

for all z in the domain and for all θ. Further, continuity also implies that even as z approaches the branch cut (in a limiting sense),

$$-2\pi \le \arg_\theta(z) \le 2\pi. \qquad (16)$$

Equations (15) and (16) along with the assumption $\Re(\beta) > 0$ give the bound

$$\Re(\beta\log_\theta(z)) = \Re(\beta)\ln|z| - \Im(\beta)\arg_\theta(z) \le \Re(\beta)\ln|z| + 2\pi|\Im(\beta)|. \qquad (17)$$

Since $|e^z| = e^{\Re(z)}$, we can now bound

$$\begin{aligned}
|m_{\alpha,\beta,\theta}(z)| &= \frac{|e^{\beta\log_\theta(z)}|}{|z-\alpha|},\\
&= \frac{e^{\Re(\beta\log_\theta(z))}}{|z-\alpha|}, \qquad (18)\\
&\le \frac{e^{\Re(\beta)\ln|z|+2\pi|\Im(\beta)|}}{||z|-|\alpha||}. \qquad (19)
\end{aligned}$$

For $t\in[\theta,\theta+2\pi]$, the bound (19) immediately gives

$$|m_{\alpha,\beta,\theta}(e^{it})\, ie^{it}| \le \frac{e^{\Re(\beta)\ln|e^{it}|+2\pi|\Im(\beta)|}}{||e^{it}|-|\alpha||} = \frac{e^{2\pi|\Im(\beta)|}}{|1-|\alpha||}. \qquad (20)$$

Thus both limits on the right hand side of (14) converge, and hence the equality in (12) is justified.

Now we show that the portion of the integral over the contour $D_{\varepsilon,\theta,\rho}$ approaches 0 as $\varepsilon \to 0^+, \rho \to 0^+$. Using (19) and applying an ML-bound to (7) yields

$$\begin{aligned}\left|\int_{D_{\varepsilon,\theta,\rho}} m_{\alpha,\beta,\theta}(z)\,dz\right| &= \left|\int_{\theta+2\pi-\varepsilon}^{\theta+\varepsilon} m_{\alpha,\beta,\theta}(\rho e^{it})\; i\rho e^{it}\,dt\right|,\\ &\leq \rho\,\frac{e^{\Re(\beta)\ln|\rho|+2\pi|\Im(\beta)|}}{|\rho-|\alpha||}(2\pi-2\varepsilon),\\ &\leq 2\pi e^{2\pi|\Im(\beta)|}\frac{\rho^{\Re(\beta)}}{|\frac{|\alpha|}{\rho}-1|}. \end{aligned} \tag{21}$$

Since $|\alpha| > 0$, (21) gives

$$\begin{aligned}\left|\lim_{\rho\to 0^+}\lim_{\varepsilon\to 0^+}\int_{D_{\varepsilon,\theta}} m_{\alpha,\beta,\theta}(z)\,dz\right| &\leq \lim_{\rho\to 0^+}\lim_{\varepsilon\to 0^+}\left[2\pi e^{2\pi|\Im(\beta)|}\frac{\rho^{\Re(\beta)}}{|\frac{|\alpha|}{\rho}-1|}\right],\\ &= \lim_{\rho\to 0^+}\left[2\pi e^{2\pi|\Im(\beta)|}\frac{\rho^{\Re(\beta)}}{|\frac{|\alpha|}{\rho}-1|}\right],\\ &= 0. \end{aligned} \tag{22}$$

We now consider the limiting value of the integral along $L_{\varepsilon,\theta,\rho}$. The core difficulty of this part of the contour integral is in evaluating

$$\lim_{\varepsilon\to 0^+}\int_{\rho}^{1} m_{\alpha,\beta,\theta}(te^{i(\theta+\varepsilon)})e^{i(\theta+\varepsilon)}\,dt. \tag{23}$$

The strategy is to use Lebesgue's Dominated Convergence Theorem (see §7, specifically Theorem 2). Take the family of functions defined on $t \in [0,1]$:

$$\mathscr{F}_L := \left\{ f_\varepsilon(t) = m_{\alpha,\beta,\theta}(te^{i(\theta+\varepsilon)})e^{i(\theta+\varepsilon)} \;\middle|\; 0 < \varepsilon < \pi \right\}. \tag{24}$$

The function $te^{i(\theta+\varepsilon)}$ is continuous in the positive real variable t, and $m_{\alpha,\beta,\theta}$ is continuous on its simply connected domain except at the point α; this singularity is undesirable. To achieve continuity of the functions in question, we restrict our view to the following collection instead:

Fix r with $0 < r < \min_{\arg(\alpha)}|\arg(\alpha)-\theta|$, and define

$$\mathscr{F}_L^* := \left\{ f_\varepsilon(t) = m_{\alpha,\beta,\theta}(te^{i(\theta+\varepsilon)})e^{i(\theta+\varepsilon)} \;\middle|\; 0 < \varepsilon < r \right\}. \tag{25}$$

The point α is outside the sector between $\theta - r$ and $\theta + r$ (see Figure 2.3 for illustration), so now the functions are continuous. This allows us to conclude that $\mathscr{F}_L^*$ is a set of Lebesgue measurable functions.

Moreover, in $\mathscr{F}_L^*$, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} f_\varepsilon(t) &= \lim_{\varepsilon\to 0^+} m_{\alpha,\beta,\theta}(te^{i(\theta+\varepsilon)})e^{i(\theta+\varepsilon)}, \\
&= \lim_{\varepsilon\to 0^+} \frac{e^{\beta \log_\theta(te^{i(\theta+\varepsilon)})}}{te^{i(\theta+\varepsilon)}-\alpha}e^{i(\theta+\varepsilon)}, \\
&= \lim_{\varepsilon\to 0^+} \frac{e^{\beta(\ln t + i(\theta-2\pi+\varepsilon))}}{te^{i(\theta+\varepsilon)}-\alpha}e^{i(\theta+\varepsilon)}, \\
&= \frac{e^{\beta(\ln t+i(\theta-2\pi))}}{te^{i\theta}-\alpha}e^{i\theta} =: g_\theta(t). \qquad (26)
\end{aligned}$$

Note that $t \in [\rho,1] \subseteq (0,\infty)$ above, and since $\mathrm{Arg}(\alpha) \neq \theta$ we have that $te^{i\theta}-\alpha \neq 0$ in the limit. The limit above is evaluated using this fact along with continuity of the exponential.
Equivalently, this means that for any sequence $\varepsilon_n \to 0^+$, f_{ε_n} converges pointwise to g_θ as $n \to \infty$.

In fact, the conditions for Lebesgue's Dominated Convergence Theorem above can be shown for $\mathscr{F}_L$ rather than $\mathscr{F}_L^*$ using a slightly more advanced argument. For the last condition however, which requires us to bound functions in the family by a Lebesgue integrable function, it is much easier to consider only $\mathscr{F}_L^*$. For a given $f_\varepsilon \in \mathscr{F}_L^*$, let $\delta := \min_{\arg(\alpha)}\{|\arg(\alpha)-(\theta+\varepsilon)|\}$. That is, δ gives the minimum difference in angle between $\theta+\varepsilon$ and the vector from the origin out to α.

If $\delta \geq \frac{\pi}{2}$, simple geometry gives that α is at least a distance of $|\alpha|$ away from the segment $te^{i(\theta+\varepsilon)}$ for $t \in [\rho,1]$. To see this, consider Figure 2.2 and note that the side of

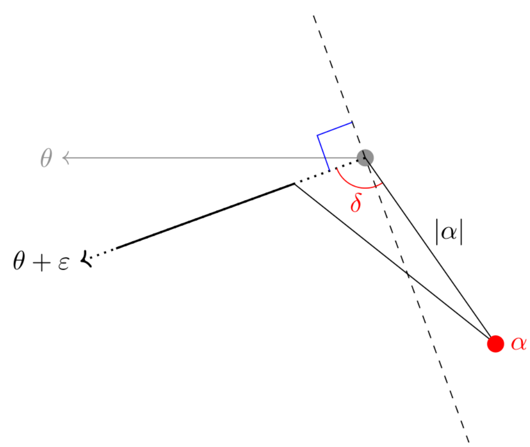

Figure 2.2: Illustration of the case $\delta \geq \frac{\pi}{2}$.

the triangle opposite the angle of size δ is the longest side of the triangle (since δ is either right or obtuse). Thus the shortest distance d from α to the line segment $L_{\varepsilon,\theta,\rho}$ is bounded below:

$$d > |\alpha|. \qquad (27)$$

If instead $\delta < \frac{\pi}{2}$, then α is at least a distance of $|\alpha|\sin(\delta)$ away from the segment $te^{i(\theta+\varepsilon)}$ for $t \in [\rho, 1]$. To see this consider the similar picture in Figure 2.3 and note

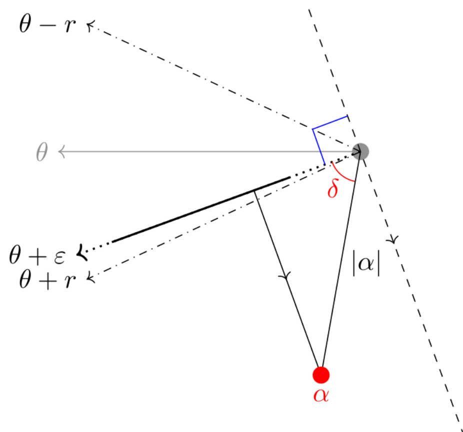

Figure 2.3: Illustration of the case $\delta < \frac{\pi}{2}$.

that the altitude dropped from α to the line containing $L_{\varepsilon,\theta,\rho}$ is precisely of length $|\alpha|\sin(\delta)$ (although the distance will be greater if $|\alpha|$ is so small or so large that the altitude dropped onto the line does not strike within the segment parameterized by $t \in [\rho, 1]$).
Now in our consideration of $\mathscr{F}_L^*$, we have $\varepsilon < r < \delta$ and so

$$|\alpha|\sin(\delta) > |\alpha|\sin\left(\min_{\arg(\alpha)}\left\{|\arg(\alpha) - (\theta \pm r)|\right\}\right) > k > 0,$$

for a fixed constant k dependent on r, θ, and α. Thus in this case as well, the shortest distance d from α to the line segment $L_{\varepsilon,\theta,\rho}$ is bounded below:

$$d > k. \tag{28}$$

Consequently, for $K := \max\{\frac{1}{|\alpha|}, \frac{1}{k}\}$, we have that every function $f_\varepsilon \in \mathscr{F}_L^*$ has for all $t \in [\rho, 1]$ that

$$\begin{aligned}
|f_\varepsilon(t)| &= |m_{\alpha,\beta,\theta}(te^{i(\theta+\varepsilon)})e^{i(\theta+\varepsilon)}|, \\
&= |m_{\alpha,\beta,\theta}(te^{i(\theta+\varepsilon)})|, \\
&= \frac{e^{\Re(\beta \log_\theta(te^{i(\theta+\varepsilon)}))}}{|te^{i(\theta+\varepsilon)} - \alpha|}, && (29) \\
&\leq Ke^{\Re(\beta)\ln|te^{i(\theta+\varepsilon)}| + 2\pi|\Im(\beta)|}, \\
&\leq Ke^{\Re(\beta)|t| + 2\pi|\Im(\beta)|} =: h_r(t). && (30)
\end{aligned}$$

(The third equality holds by (18); the first inequality holds by (17) and the reasoning which led to (27) and (28); the final inequality holds since $|t| > \ln|t|$ for all $t \in \mathbb{R}$ and the because exponential is strictly increasing on $\mathbb{R}$.)
Clearly this h_r is integrable on $[\rho, 1]$ for all $\rho > 0$, since it is simply a scaled exponential.

Now, consider any arbitrary sequence $(\varepsilon_n) \to 0^+$ with $\varepsilon_n < r$, and define a sequence of functions (f_{ε_n}); note $f_{\varepsilon_n} \in \mathscr{F}_L^*$ for all n. From (26) we have $(f_{\varepsilon_n}) \to g_\theta$ pointwise, and $|f_{\varepsilon_n}(t)| \le h_r(t)$ for all n and for all $t \in [0,1]$, as shown in (30). Therefore Lebesgue's Dominated Convergence Theorem implies that

$$\lim_{n\to\infty} \int_\rho^1 f_{\varepsilon_n}(t)\, dt = \int_\rho^1 g_\theta(t)\, dt. \tag{31}$$

for all $\rho > 0$.
Dispensing with the condition $\varepsilon_n < r$, it is still true for any arbitrary sequence $(\varepsilon_n) \to 0^+$ that

$$\lim_{n\to\infty} \int_\rho^1 f_{\varepsilon_n}(t)\, dt = \int_\rho^1 g_\theta(t)\, dt \tag{32}$$

since $(\varepsilon_n) \to 0^+$ has a tail which is completely bounded above by r, and thus convergence of the tail shown in (31) implies convergence of the whole sequence. Since (32) holds for arbitrary (ε_n), this implies

$$\lim_{\varepsilon\to 0^+} \int_\rho^1 f_\varepsilon(t)\, dt = \int_\rho^1 g_\theta(t)\, dt. \tag{33}$$

for $f_\varepsilon \in \mathscr{F}_L$.

Hence we evaluate (23) and find

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \int_\rho^1 m_{\alpha,\beta,\theta}(te^{i(\theta+\varepsilon)})e^{i(\theta+\varepsilon)}\, dt &= \int_\rho^1 \lim_{\varepsilon\to 0^+} m_{\alpha,\beta,\theta}(te^{i(\theta+\varepsilon)})e^{i(\theta+\varepsilon)} dt, \\
&= \int_\rho^1 \frac{e^{\beta(\ln t + i(\theta-2\pi))}}{te^{i\theta}-\alpha} e^{i\theta}\, dt, \\
&= e^{i\beta(\theta-2\pi)} \int_\rho^1 \frac{e^{\beta \ln t}}{t-\alpha e^{-i\theta}}\, dt.
\end{aligned} \tag{34}$$

But g_θ is continuous for $t \in (0,1]$ and bounded as $t \to 0$. Therefore, allowing improper integrals, and drawing from equations (4) and (34) it is straightforward to compute

$$\begin{aligned}
\lim_{\rho\to 0^+} \lim_{\varepsilon\to 0^+} \int_{L_{\varepsilon,\theta,\rho}} m_{\alpha,\beta,\theta}(z)\, dz &= \lim_{\rho\to 0^+} \lim_{\varepsilon\to 0^+} \int_\rho^1 m_{\alpha,\beta,\theta}(te^{i(\theta+\varepsilon)})e^{i(\theta+\varepsilon)}\, dt, \\
&= \lim_{\rho\to 0^+} \left[e^{i\beta(\theta-2\pi)} \int_\rho^1 \frac{e^{\beta \ln t}}{t-\alpha e^{-i\theta}}\, dt \right], \\
&= e^{i\beta(\theta-2\pi)} \int_0^1 \frac{e^{\beta \ln t}}{t-\alpha e^{-i\theta}}\, dt.
\end{aligned} \tag{35}$$

Finally we take limits in the last integral on the right hand side of (9) along $M_{\varepsilon,\theta,\rho}$. Similarly the difficulty in this case is evaluating

$$\lim_{\varepsilon\to 0^+} \int_1^\rho m_{\alpha,\beta,\theta}(te^{i(\theta+2\pi-\varepsilon)})e^{i(\theta+2\pi-\varepsilon)}\, dt \tag{36}$$

using Lebesgue's Dominated Convergence Theorem. Analogous steps as those used for $L_{\varepsilon,\theta,\rho}$ can be applied to the $M_{\varepsilon,\theta,\rho}$ case to show that

$$\begin{aligned}
\lim_{\varepsilon\to 0^+}\int_1^{\rho} m_{\alpha,\beta,\theta}(te^{i(\theta+2\pi-\varepsilon)})e^{i(\theta+2\pi-\varepsilon)}\,dt &= -\int_{\rho}^{1}\lim_{\varepsilon\to 0^+} m_{\alpha,\beta,\theta}(te^{i(\theta+2\pi-\varepsilon)})e^{i(\theta+2\pi-\varepsilon)}\,dt,\\
&= -\int_{\rho}^{1}\frac{e^{\beta(\ln(t)+i\theta)}}{te^{i(\theta+2\pi)}-\alpha}e^{i(\theta+2\pi)}\,dt,\\
&= -e^{i\beta\theta}\int_{\rho}^{1}\frac{e^{\beta\ln t}}{t-\alpha e^{-i\theta}}\,dt. \qquad (37)
\end{aligned}$$

Just as before the integrand is bounded on $[0,1]$. Using (37) there is no issue writing

$$\begin{aligned}
\lim_{\rho\to 0^+}\lim_{\varepsilon\to 0^+}\int_{M_{\varepsilon,\theta,\rho}} m_{\alpha,\beta,\theta}(z)\,dz &= \lim_{\rho\to 0^+}\lim_{\varepsilon\to 0^+}\int_1^{\rho} m_{\alpha,\beta,\theta}(te^{i(\theta+2\pi-\varepsilon)})e^{i(\theta+2\pi-\varepsilon)}\,dt,\\
&= \lim_{\rho\to 0^+}\left[-e^{i\beta\theta}\int_{\rho}^{1}\frac{e^{\beta\ln t}}{t-\alpha e^{-i\theta}}\,dt\right],\\
&= -e^{i\beta\theta}\int_0^1\frac{e^{\beta\ln t}}{t-\alpha e^{-i\theta}}\,dt. \qquad (38)
\end{aligned}$$

3 The Hypergeometric Function Connection

3.1 A Core Integral

In order to fully evaluate (1) using the contour method outlined in §2, the following integral from Lemma 1 must be evaluated:

$$\int_0^1\frac{e^{\beta\ln t}}{t-\alpha e^{-i\theta}}\,dt, \qquad (39)$$

which exists for $\Re(\beta) > -1$. The integral in (39) is in fact an improper integral and can be written

$$\lim_{\rho\to 0}\int_{\rho}^{1}\frac{e^{\beta\ln t}}{t-\alpha e^{-i\theta}}\,dt.$$

Algebraic manipulations give

$$\begin{aligned}
\frac{e^{\beta\ln t}}{t-\alpha e^{-i\theta}} &= \frac{e^{\ln t}}{t-\alpha e^{-i\theta}}e^{(\beta-1)\ln t},\\
&= \frac{t-\alpha e^{-i\theta}+\alpha e^{-i\theta}}{t-\alpha e^{-i\theta}}e^{(\beta-1)\ln t},\\
&= \left(1+\frac{\alpha e^{-i\theta}}{t-\alpha e^{-i\theta}}\right)e^{(\beta-1)\ln t}. \qquad (40)
\end{aligned}$$

Integrating first over the interval $[\rho,1]$ and using (40) yields

$$\int_{\rho}^{1}\frac{e^{\beta\ln t}}{t-\alpha e^{-i\theta}}\,dt = \int_{\rho}^{1}e^{(\beta-1)\ln t}\,dt + \alpha e^{-i\theta}\int_{\rho}^{1}\frac{e^{(\beta-1)\ln t}}{t-\alpha e^{-i\theta}}\,dt. \qquad (41)$$

Notice that the function $e^{(\beta-1)\log(t)}$ is the derivative of $\frac{1}{\beta}e^{\beta\log(t)}$, which is analytic on $[\rho,1]$. Thus

$$\int_\rho^1 e^{(\beta-1)\ln t}\,dt = \frac{1}{\beta}e^{\beta\ln(1)} - \frac{1}{\beta}e^{\beta\ln(\rho)} = \frac{1}{\beta} - \frac{1}{\beta}e^{\beta\ln(\rho)}. \tag{42}$$

Substituting (42) into (41) and taking limits gives

$$\lim_{\rho\to 0}\int_\rho^1 \frac{e^{\beta\ln t}}{t-\alpha e^{-i\theta}}\,dt = \lim_{\rho\to 0}\left[\frac{1}{\beta} - \frac{1}{\beta}e^{\beta\ln(\rho)}\right] + \alpha e^{-i\theta}\lim_{\rho\to 0}\int_\rho^1 \frac{e^{(\beta-1)\ln t}}{t-\alpha e^{-i\theta}}\,dt,$$

$$\int_0^1 \frac{e^{\beta\ln t}}{t-\alpha e^{-i\theta}}\,dt = \frac{1}{\beta} + \alpha e^{-i\theta}\int_0^1 \frac{e^{(\beta-1)\ln t}}{t-\alpha e^{-i\theta}}\,dt; \tag{43}$$

the integral on the right hand side of (43) exists for $\Re(\beta) > 0$. Again the convergence of improper integrals follows from the boundedness of the integrands. Moving the constant inside the integral in (43) gives

$$\int_0^1 \frac{e^{\beta\ln t}}{t-\alpha e^{-i\theta}}\,dt = \frac{1}{\beta} - \int_0^1 \frac{e^{(\beta-1)\ln t}}{1-\left(\frac{1}{\alpha}e^{i\theta}\right)t}\,dt. \tag{44}$$

Therefore finding a solution to (1) using the contour integration method necessitates working with the following "core integral" for $z = \frac{1}{\alpha}e^{i\theta}$:

$$\int_0^1 t^{\beta-1}(1-zt)^{-1}\,dt. \tag{45}$$

The choice to write $t^{\beta-1}$ rather than $e^{(\beta-1)\ln t}$ in (45) is intentional, since generality is not lost when any branch $\log_\theta$ for $\theta \not\equiv 0$ is used to define this complex power of $t \in [0,1]$.

3.2 Definition & Relevant Identities

We investigate the integral in (45) by making use of the well-studied hypergeometric function ${}_2F_1(a,b,c;z)$. For $|z| < 1$, this function is defined as the infinite series

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!}z^n, \quad c \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0} \tag{46}$$

where $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$ is the rising Pochhammer symbol.

The hypergeometric series generalizes the geometric series, and is prominent in the study of linear differential equations with three regular singular points. The hypergeometric function is notably a solution to the hypergeometric equation, discussed in §6.

A comprehensive collection of identities involving ${}_2F_1$ can be found in [2]. The most notable for our purposes is the following:

For $|z| < 1$ and $\Re(c) > \Re(b) > 0$,

$${}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt. \tag{47}$$

Letting $a = 1$, $b = \beta$, and $c = 1+\beta$ under the conditions for (47) gives

$$
\begin{aligned}
{}_2F_1(1,\beta,1+\beta;z) &= \frac{\Gamma(1+\beta)}{\Gamma(\beta)\Gamma(1)} \int_0^1 t^{\beta-1}(1-t)^0(1-tz)^{-1}\,dt, \\
&= \beta \int_0^1 t^{\beta-1}(1-zt)^{-1}\,dt, \qquad (48)
\end{aligned}
$$

such that the integral above is exactly the integral in (45), only scaled.

3.3 Final Steps of the Contour Method

We conclude the contour method for $|\alpha| > 1$ by proving the following statement, making use of the hypergeometric identity (48).

Proposition 1. *When $|\alpha| > 1$, $\mathrm{Arg}(\alpha) \neq \theta$, $\theta \neq 0$ (mod 2π), and $\Re(\beta) > 0$,*

$$
\int_{\partial\mathbb{D}} m_{\alpha,\beta,\theta} = e^{i\beta\theta}(1-e^{-2\pi i\beta})\frac{1}{\beta}\left[1 - {}_2F_1(1,\beta;1+\beta;\alpha^{-1}e^{i\theta})\right]. \qquad (49)
$$

Proof. Since $|\alpha| > 1$, the last argument in the hypergeometric function satisfies

$$
\left|\frac{1}{\alpha}e^{i\theta}\right| = \frac{1}{|\alpha|} < 1.
$$

Under the assumption $\Re(\beta) > 0$, one can apply the identity (48) and find

$$
\int_0^1 \frac{t^{\beta-1}}{1-(\alpha^{-1}e^{i\theta})t}\,dt = \frac{1}{\beta}\,{}_2F_1(1,\beta;1+\beta;\alpha^{-1}e^{i\theta}). \qquad (50)
$$

With this expression for the core integral, an application of Lemma 1 and (44) completes the proof. □

The case $0 < |\alpha| < 1$ cannot be approached in the same manner. While the initial steps in the contour method still hold, the integral identity from (48) does not apply since the last argument in the hypergeometric function now satisfies

$$
\left|\frac{1}{\alpha}e^{i\theta}\right| = \frac{1}{|\alpha|} > 1;
$$

which is outside the domain of (47).

4 Series Method

Fortunately there exist methods outside of contour integration which allow us to express (1) in terms of the hypergeometric function in all cases. Rather than dealing with integral identities of the hypergeometric function, one can work with series to produce a term of the form (46).

4.1 Proof of the main result

Consider first $|\alpha| > 1$. Recall from (12) and (13) that

$$\int_{\partial\mathbb{D}} m_{\alpha,\beta,\theta}(z)\,dz = \int_{|z|=1} \frac{e^{\beta \log_\theta(z)}}{z-\alpha}\,dz,$$
$$= \lim_{\varepsilon\to 0^+} \int_{\theta-2\pi+\varepsilon}^{\theta-\varepsilon} m_{\alpha,\beta,\theta}(e^{it}) ie^{it}\,dt. \tag{51}$$

For $\theta - 2\pi < t < \theta$, $\log_\theta(e^{it}) = it$, so one can then rewrite the integrand as

$$\begin{aligned} m_{\alpha,\beta,\theta}(e^{it})ie^{it} &= \frac{e^{\beta(\log_\theta(e^{it}))}}{e^{it}-\alpha} ie^{\log_\theta(e^{it})}, \\ &= i\frac{e^{(\beta+1)(\log_\theta(e^{it}))}}{e^{it}-\alpha}, \\ &= i\frac{e^{i(\beta+1)t}}{e^{it}-\alpha}, \\ &= -\frac{ie^{i(\beta+1)t}}{\alpha}\cdot\frac{1}{1-\frac{1}{\alpha}e^{it}}, \\ &= -\frac{ie^{i(\beta+1)t}}{\alpha}\sum_{k=0}^{\infty}\alpha^{-k}e^{ikt}; \end{aligned} \tag{52}$$

where (52) follows by rewriting in terms of a convergent geometric series. Pulling the factor of e^{it} inside the series yields

$$\begin{aligned} m_{\alpha,\beta,\theta}(e^{it})ie^{it} &= -ie^{i\beta t}\sum_{k=0}^{\infty}\alpha^{-(k+1)}e^{i(k+1)t}, \\ &= -ie^{i\beta t}\sum_{k=1}^{\infty}\alpha^{-k}e^{ikt}, \\ &= -i\sum_{k=1}^{\infty}\alpha^{-k}e^{i(\beta+k)t}. \end{aligned} \tag{53}$$

For a fixed $|\alpha| > 1$ we have that $|\alpha|^{-1} < 1$, so

$$\left|\sum_{k=1}^{\infty}\alpha^{-k}e^{ikt}\right| \le \sum_{k=1}^{\infty}|\alpha^{-k}e^{ikt}| = \sum_{k=1}^{\infty}|\alpha|^{-k} =: K_\alpha < \infty.$$

We define a sequence of functions (f_n), where $f_n : [\theta-2\pi,\theta] \to \mathbb{C}$ are given by

$$f_n(t) := \sum_{k=1}^{n}\alpha^{-k}e^{i(\beta+k)t} = e^{i\beta t}\sum_{k=1}^{n}\alpha^{-k}e^{ikt}.$$

Each function in the sequence is bounded via

$$\begin{aligned} |f_n(t)| &= \left|e^{i\beta t}\sum_{k=1}^{n}\alpha^{-k}e^{ikt}\right|, \\ &\le K_\alpha|e^{i\beta t}|, \\ &= K_\alpha e^{-\Im(\beta)t} =: g_\alpha(t). \end{aligned}$$

Note that g_α is integrable on $[\theta-2\pi+\varepsilon, \theta-\varepsilon]$ since it is simply a scaled exponential. It is clear that each f_n is continuous as a finite sum of analytic functions, so again this continuity means the functions are measurable. Since their pointwise limit is the expression in (53), Lebesgue's Dominated Convergence Theorem (see §7, Theorem 2 specifically) implies

$$\begin{aligned}\int_{\theta-2\pi+\varepsilon}^{\theta-\varepsilon}\sum_{k=1}^{\infty}\alpha^{-k}e^{i(\beta+k)t}\,dt &= \lim_{n\to\infty}\int_{\theta-2\pi+\varepsilon}^{\theta-\varepsilon}\sum_{k=1}^{n}\alpha^{-k}e^{i(\beta+k)t}\,dt,\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\int_{\theta-2\pi+\varepsilon}^{\theta-\varepsilon}\alpha^{-k}e^{i(\beta+k)t}\,dt,\\ &= \sum_{k=1}^{\infty}\int_{\theta-2\pi+\varepsilon}^{\theta-\varepsilon}\alpha^{-k}e^{i(\beta+k)t}\,dt, \qquad (54)\end{aligned}$$

where the second equality holds since it is merely the interchange of an integral and finite sum.

Using (54) along with (53) yields

$$\begin{aligned}\int_{\theta-2\pi+\varepsilon}^{\theta-\varepsilon} m_{\alpha,\beta,\theta}(e^{it})ie^{it}\,dt &= -i\int_{\theta-2\pi+\varepsilon}^{\theta-\varepsilon}\sum_{k=1}^{\infty}\alpha^{-k}e^{i(\beta+k)t}\,dt,\\ &= -i\sum_{k=1}^{\infty}\int_{\theta-2\pi+\varepsilon}^{\theta-\varepsilon}\alpha^{-k}e^{i(\beta+k)t}\,dt,\\ &= -i\sum_{k=1}^{\infty}\alpha^{-k}\int_{\theta-2\pi+\varepsilon}^{\theta-\varepsilon}e^{i(\beta+k)t}\,dt. \qquad (55)\end{aligned}$$

An individual summand of (55) consists of an α^{-k} term multiplied by an integral. The integrand, $e^{i(\beta+k)t}$, is an entire function of t and thus is bounded on the compact set $t\in[\theta-2\pi,\theta]$ by some M. Note that this bound M can be chosen independent of k by letting

$$M > e^{-\Im(\beta)t} = |e^{i(\beta+k)t}| \qquad \forall t\in[\theta-2\pi,\theta].$$

The length of the curve being integrated over is at most

$$(\theta-\varepsilon)-(\theta-2\pi+\varepsilon) = 2\pi-2\varepsilon < 2\pi =: L,$$

where L does not depend on ε. Because the integrand is entire it must be continuous on the path of integration, and so the ML-bound gives that

$$\left|\int_{\theta-2\pi+\varepsilon}^{\theta-\varepsilon}e^{i(\beta+k)t}\,dt\right| \le ML,$$

where M and L are given above and independent of ε and k. Thus each term of the series in (55) is bounded in modulus by $ML|\alpha|^{-k}$, so that

$$\left|\sum_{k=1}^{\infty}\alpha^{-k}\int_{\theta-2\pi+\varepsilon}^{\theta-\varepsilon}e^{i(\beta+k)t}\,dt\right| \le \sum_{k=1}^{\infty}ML|\alpha|^{-k}. \qquad (56)$$

Since $|\alpha|>1$, the right hand side of (56) converges, and the Weierstrass M-test implies that the series in (55) is uniformly convergent. Substituting the expression from (55) back into (51) allows the exchange of limit and infinite sum in (57) to find that

$$\int_{\partial\mathbb{D}} m_{\alpha,\beta,\theta}(z)\,dz = \lim_{\varepsilon\to 0^+}\left[-i\sum_{k=1}^{\infty}\alpha^{-k}\int_{\theta-2\pi+\varepsilon}^{\theta-\varepsilon} e^{i(\beta+k)t}\,dt\right],$$

$$= -i\sum_{k=1}^{\infty}\alpha^{-k}\lim_{\varepsilon\to 0^+}\int_{\theta-2\pi+\varepsilon}^{\theta-\varepsilon} e^{i(\beta+k)t}\,dt, \tag{57}$$

$$= -i\sum_{k=1}^{\infty}\alpha^{-k}\int_{\theta-2\pi}^{\theta} e^{i(\beta+k)t}\,dt. \tag{58}$$

The integrand in (58) is entire, and it has an antiderivative $\frac{e^{i(\beta+k)t}}{i(\beta+k)}$ when $\beta+k\neq 0$; this antiderivative is also entire. This fact not only ensures the equality between (57) and (58), but it also allows the use of the Complex Fundamental Theorem of Calculus to conclude that for $\beta\notin\mathbb{Z}_{<0}$,

$$\int_{\theta-2\pi}^{\theta} e^{i(\beta+k)t}\,dt = \left[\frac{e^{i(\beta+k)t}}{i(\beta+k)}\right]_{\theta-2\pi}^{\theta} = \frac{e^{i(\beta+k)\theta}}{i(\beta+k)}\left(1-e^{-2\pi i\beta}\right). \tag{59}$$

Using definition (46) as well as our intermediates (58) and (59) we find

$$\int_{\partial\mathbb{D}} m_{\alpha,\beta,\theta}(z)\,dz = -i\sum_{k=1}^{\infty}\alpha^{-k}\frac{e^{i(\beta+k)\theta}}{i(\beta+k)}\left(1-e^{-2\pi i\beta}\right),$$

$$= -e^{i\beta\theta}\left(1-e^{-2\pi i\beta}\right)\sum_{k=1}^{\infty}\alpha^{-k}\frac{e^{ik\theta}}{\beta+k},$$

$$= -e^{i\beta\theta}\left(1-e^{-2\pi i\beta}\right)\frac{1}{\beta}\left[\left(\sum_{k=0}^{\infty}\frac{\beta}{\beta+k}(\alpha^{-1}e^{i\theta})^k\right)-1\right],$$

$$= -e^{i\beta\theta}\left(1-e^{-2\pi i\beta}\right)\frac{1}{\beta}\left[\left(\sum_{k=0}^{\infty}\frac{(1)_k(\beta)_k}{(1+\beta)_k k!}(\alpha^{-1}e^{i\theta})^k\right)-1\right],$$

$$= -e^{i\beta\theta}\left(1-e^{-2\pi i\beta}\right)\frac{1}{\beta}\left[{}_2F_1(1,\beta;1+\beta;\alpha^{-1}e^{i\theta})-1\right],$$

$$= e^{i\beta\theta}\left(1-e^{-2\pi i\beta}\right)\frac{1}{\beta}\left[1-{}_2F_1(1,\beta;1+\beta;\alpha^{-1}e^{i\theta})\right], \tag{60}$$

so long as $\beta\neq 0$. This completes the proof of Theorem 1 in the case where $\beta\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0}$. To handle the cases when $\beta\in\mathbb{Z}_{\leq 0}$, note that

$$\int_{\theta-2\pi}^{\theta} e^{i(\beta+k)t}\,dt = \begin{cases} 2\pi & \beta+k=0,\\ 0 & \beta+k\in\mathbb{Z}\setminus\{0\},\end{cases} \tag{61}$$

since the bounds of integration align with the period of the exponential unless the exponent is 0. Thus when $\beta\in\mathbb{Z}_{\leq 0}$,

$$\int_{\partial\mathbb{D}} m_{\alpha,\beta,\theta}(z)\,dz = -i\sum_{k=1}^{\infty}\alpha^{-k}2\pi\delta_{\beta,-k} \tag{62}$$

where $\delta_{\beta,-k}$ is the classical Kronecker delta function. This completes the proof of Theorem 1 in the case where $\beta\in\mathbb{Z}_{\leq 0}$, the first part of the main result.

Next consider $|\alpha|<1$. We proceed in a manner analogous to that of the $|\alpha|>1$ case, omitting details for the sake of brevity. It holds that

$$\int_{\partial\mathbb{D}} m_{\alpha,\beta,\theta} = \begin{cases} e^{i\beta\theta}\left(1-e^{-2\pi i\beta}\right)\sum_{k=0}^{\infty}\frac{\alpha^k}{\beta-k}e^{-ik\theta} & \beta\notin\mathbb{Z}_{\geq 0}, \\ 2\pi i\alpha^{\beta} & \beta\in\mathbb{Z}_{\geq 0}. \end{cases} \tag{63}$$

An application of (46) shows that for $\beta \neq 0$,

$$\begin{aligned} \sum_{k=0}^{\infty}\frac{\alpha^k}{\beta-k}e^{-ik\theta} &= \frac{1}{\beta}\sum_{k=0}^{\infty}\frac{-\beta}{-\beta+k}\left(\alpha e^{-i\theta}\right)^k, \\ &= \frac{1}{\beta}\sum_{k=0}^{\infty}\frac{(1)_k(-\beta)_k}{(1-\beta)_k k!}\left(\alpha e^{-i\theta}\right)^k, \\ &= \frac{1}{\beta}\,{}_2F_1(1,-\beta;1-\beta;\alpha e^{-i\theta}). \end{aligned} \tag{64}$$

Combining (63) and (64) completes the proof of Theorem 1.

4.2 Reconciling Methods

Note that the steps of the contour method described in §2 and the simplifications in §3.1 still hold so long as $\mathrm{Arg}(\alpha)\neq\theta$, $\Re(\beta)>0$, $\theta\neq 0 \pmod{2\pi}$, and $|\alpha|\neq 0,1$. The nontrivial equations of the series method hold so long as $|\alpha|\neq 1$ and $\beta\notin\mathbb{Z}_{\geq 0}$. Thus under all these conditions one can write an identity for (45) in the case where $0<|\alpha|<1$:

$$e^{i\beta\theta}\left(1-e^{-2\pi i\beta}\right)\frac{1}{\beta}\,{}_2F_1(1,-\beta;1-\beta;\alpha e^{-i\theta}) = 2\pi i\alpha^{\beta} + e^{i\beta\theta}\left(1-e^{-2\pi i\beta}\right)\int_0^1 \frac{e^{\beta\ln t}}{t-\alpha e^{-i\theta}}\,dt, \tag{65}$$

$$\frac{1}{\beta}\,{}_2F_1(1,-\beta;1-\beta;\alpha e^{-i\theta}) = \frac{2\pi i\alpha^{\beta}}{e^{i\beta\theta}\left(1-e^{-2\pi i\beta}\right)} + \int_0^1 \frac{e^{\beta\ln t}}{t-\alpha e^{-i\theta}}\,dt,$$

$$\frac{1}{\beta}\,{}_2F_1(1,-\beta;1-\beta;\alpha e^{-i\theta}) = \frac{2\pi i\alpha^{\beta}}{e^{i\beta\theta}\left(1-e^{-2\pi i\beta}\right)} + \frac{1}{\beta} - \int_0^1 \frac{e^{(\beta-1)\ln t}}{1-\left(\frac{1}{\alpha}e^{i\theta}\right)t}\,dt, \tag{66}$$

$$\int_0^1 \frac{e^{(\beta-1)\ln t}}{1-\left(\frac{1}{\alpha}e^{i\theta}\right)t}\,dt = \frac{2\pi i\alpha^{\beta}}{e^{i\beta\theta}\left(1-e^{-2\pi i\beta}\right)} + \frac{1}{\beta}\left[1-{}_2F_1(1,-\beta;1-\beta;\alpha e^{-i\theta})\right] \tag{67}$$

where the equality in (65) follows from Lemma 1 and (64), and the equality in (66) follows from (44).

5 Computing the Example $\beta = \frac{m}{n}$

Since the hypergeometric function gives the value of (1) as an infinite series which is still difficult to explicitly evaluate, it is desirable to compute examples for which the hypergeometric function can be simplified more. We show this is the case when $\beta = \frac{m}{n}\in\mathbb{Q}\setminus\mathbb{Z}$, with $m\in\mathbb{Z}$, $n\in\mathbb{N}$. We ignore the cases $\beta\in\mathbb{Z}$ since these are easily evaluated without need of the hypergeometric function.

Corollary 1. *Let $\beta = \frac{m}{n} \in \mathbb{Q} \setminus \mathbb{Z}$. When $|\alpha| < 1$,*

$$\int_{\partial\mathbb{D}} m_{\alpha,\beta,\theta} = \alpha^{-\frac{m}{n}} (1 - e^{-2\pi i \frac{m}{n}}) \sum_{j=0}^{n-1} e^{\frac{2\pi i j m}{n}} \log\left(1 - e^{\frac{2\pi i j}{n}} \sqrt[n]{\alpha e^{-i\theta}}\right)$$

and when $|\alpha| > 1$

$$\int_{\partial\mathbb{D}} m_{\alpha,\beta,\theta} = (1 - e^{-2\pi i \frac{m}{n}}) \left(\frac{n}{m} e^{i\frac{m}{n}\theta} + \alpha^{\frac{m}{n}} \sum_{j=0}^{n-1} e^{-\frac{2\pi i j m}{n}} \log\left(1 - e^{\frac{2\pi i j}{n}} \sqrt[n]{\alpha^{-1} e^{i\theta}}\right) \right).$$

Proof. From Theorem 1, one has for $\beta \notin \mathbb{Z}$ that

$$\int_{\partial\mathbb{D}} m_{\alpha,\frac{m}{n},\theta} = \begin{cases} e^{i\frac{m}{n}\theta}\left(1 - e^{-2\pi i \frac{m}{n}}\right) \frac{n}{m}\, {}_2F_1(1, -\frac{m}{n}; 1 - \frac{m}{n}; \alpha e^{-i\theta}) & |\alpha| < 1, \\ e^{i\frac{m}{n}\theta}\left(1 - e^{-2\pi i \frac{m}{n}}\right) \frac{n}{m}\left[1 - {}_2F_1(1, +\frac{m}{n}; 1 + \frac{m}{n}; \alpha^{-1} e^{i\theta})\right] & |\alpha| > 1. \end{cases} \tag{68}$$

Therefore the main difficulty in evaluating (68) lies in computing

$${}_2F_1\left(1, \frac{m}{n}; 1 + \frac{m}{n}; z\right) \tag{69}$$

for non-integral $\frac{m}{n}$ and for $0 < |z| < 1$.

Since $\frac{m}{n} \notin \mathbb{Z}$, it is never the case that the parameter $c = 1 + \frac{m}{n}$ in (69) is 0 or a negative integer. The hypergeometric series is therefore well defined, and using the definition (46) yields

$$\begin{aligned} {}_2F_1\left(1, \frac{m}{n}; 1 + \frac{m}{n}; z\right) &:= \sum_{k=0}^{\infty} \frac{(1)_k (\frac{m}{n})_k}{k!(1 + \frac{m}{n})_k} z^k. \\ &= \sum_{k=0}^{\infty} \frac{\frac{m}{n}}{k + \frac{m}{n}} z^k \\ &= m \sum_{k=0}^{\infty} \frac{1}{m + nk} z^k \\ &= \frac{m}{z^{\frac{m}{n}}} \sum_{k=0}^{\infty} \frac{1}{m + nk} z^{\frac{m}{n} + k} =: G(z). \end{aligned} \tag{70}$$

Note also that since $0 < |z|$, division by a fractional power of z causes no issue. The particular choice of branch for defining the n^{th} root does not matter so long as the choice is consistent across the fractional powers (see remarks 1 and 2).

Notice that the expression for $G(z)$ produces an even simpler expression for $G(z^n)$, given by

$$G(z^n) = \frac{m}{z^m} \sum_{k=0}^{\infty} \frac{1}{m + nk} z^{m+nk} \tag{71}$$

On the other hand, for any branch of the logarithm with $\log(1) = 0$ analytic in a ball of radius 1 at $z = 1$, we have

$$\log(1 - z) = \sum_{k=1}^{\infty} -\frac{1}{k} z^k$$

whenever $|z| < 1$. The difference between the above expression and that in (71) is that only terms of the form z^{m+nk} for $k \in \mathbb{N} \cup \{0\}$ appear in (71), whereas a z^k term appears in the series for $\log(1 - z)$ for every $k \in \mathbb{N}$. To rectify this we express $G(z^n)$ as some series $\sum_{s=1}^{\infty} \frac{\delta_s}{s} z^s$, possibly with leading factors, where δ_s takes on the value 1 whenever n divides $s-m$ (so that s is of the form $m+nk$ for some $k \in \mathbb{N}$) and is 0 otherwise.

To find a suitable function δ_s, recall that the sum of all of the n^{th} roots of unity is 0 for $n > 1$. It is natural then that

$$\delta_s = \frac{1}{n} \sum_{j=0}^{n-1} e^{\frac{2\pi i j(s-m)}{n}} = \begin{cases} 1 & \text{if } n \mid (s-m) \\ 0 & \text{otherwise} \end{cases} \tag{72}$$

is the desired function. To see the validity of this claim, we first consider when $n \mid (s-m)$. In this case, we have $\frac{s-m}{n} \in \mathbb{Z}$, and so

$$\frac{1}{n} \sum_{j=0}^{n-1} e^{\frac{2\pi i j(s-m)}{n}} = \frac{1}{n} \sum_{j=0}^{n-1} 1 = 1 \tag{73}$$

since $\frac{j(s-m)}{n} \in \mathbb{Z}$. On the other hand, when $n \nmid (s-m)$, let $d := \gcd(n, s-m)$ and define $\eta := \frac{n}{d}$. Note that $d < n$ else we have $n \mid (s-m)$, and thus $\eta > 1$. Now

$$e^{\frac{2\pi i(s-m)}{n}} = \zeta_\eta^{\frac{s-m}{d}} \tag{74}$$

where ζ_η is the first primitive η^{th} root of unity. Note also that because d is the greatest common divisor of $s-m$ and n, then $\frac{s-m}{d}$ is coprime to η. This implies that $\zeta_\eta^{\frac{s-m}{d}}$ is another primitive η^{th} root of unity. Now

$$\begin{aligned}
\delta_s &= \frac{1}{n} \sum_{j=0}^{n-1} e^{\frac{2\pi i j(s-m)}{n}}, \\
&= \frac{1}{d\eta} \sum_{j=0}^{d\eta-1} \zeta_\eta^{\frac{s-m}{d} j}, \\
&= \frac{1}{d\eta} \left(\sum_{j=0}^{\eta-1} \zeta_\eta^{\frac{s-m}{d} j} + \sum_{j=\eta}^{2\eta-1} \zeta_\eta^{\frac{s-m}{d} j} + \cdots + \sum_{j=(d-1)\eta}^{d\eta-1} \zeta_\eta^{\frac{s-m}{d} j} \right), && (75) \\
&= \frac{1}{\eta} \sum_{j=0}^{\eta-1} \zeta_\eta^{\frac{s-m}{d} j}, && (76) \\
&= 0. && (77)
\end{aligned}$$

The equality between (75) and (76) holds since $\zeta_\eta^{\frac{s-m}{d} j} = \zeta_\eta^{\frac{s-m}{d} j'}$ when $j \equiv j' \pmod{\eta}$. The final equality, (77), holds since the sum over every η^{th} root of 1 is 0.

One can therefore express

$$G(z^n) = \frac{m}{z^m} \sum_{s=1}^{\infty} \frac{\delta_s}{s} z^s \tag{78}$$

since this series gives the same terms as the series in (71). To evaluate this series, note that for each j in $\{0, \dots, n-1\}$, the series $\sum_{s=1}^{\infty} \left| \frac{e^{\frac{2\pi i j(s-m)}{n}}}{s} \right| z^s$ is a power series which converges absolutely for $|z| < 1$. Letting the value of this series be denoted b_j, we also note that $\sum_{j=0}^{n-1} b_j$ converges since it is a finite sum. We may therefore exchange the order of summation to get

$$\sum_{j=0}^{n-1}\sum_{s=1}^{\infty}\frac{e^{\frac{2\pi ij(s-m)}{n}}}{s}z^s=\sum_{s=1}^{\infty}\sum_{j=0}^{n-1}\frac{e^{\frac{2\pi ij(s-m)}{n}}}{s}z^s=G(z^n) \tag{79}$$

which allows evaluation of $G(z^n)$ by simplifying the left hand side of (79):

$$\begin{aligned}G(z^n)=\frac{m}{z^m}\sum_{j=0}^{n-1}\sum_{s=1}^{\infty}\frac{e^{\frac{2\pi ij(s-m)}{n}}}{s}z^s &= \frac{m}{z^m}\sum_{j=0}^{n-1}e^{-\frac{2\pi ijm}{n}}\sum_{s=1}^{\infty}\frac{e^{\frac{2\pi ijs}{n}}}{s}z^s,\\ &= \frac{m}{z^m}\sum_{j=0}^{n-1}-e^{-\frac{2\pi ijm}{n}}\sum_{s=1}^{\infty}-\frac{1}{s}(e^{\frac{2\pi ij}{n}}z)^s,\\ &= \frac{m}{z^m}\sum_{j=0}^{n-1}-e^{-\frac{2\pi ijm}{n}}\log(1-e^{\frac{2\pi ij}{n}}z). \end{aligned}\tag{80}$$

Remark 1. Notice that the only requirement of the branch of log we choose is that it is analytic in the ball of radius 1 at $z=1$, and that $\log(1)=0$.

Finally, to come up with an expression for $G(z)$ as opposed to $G(z^n)$, simply substitute $z^{\frac{1}{n}}$ in the expression above, yielding

$$G(z)=-\frac{m}{n}z^{-\frac{m}{n}}\sum_{j=0}^{n-1}e^{-\frac{2\pi ijm}{n}}\log(1-e^{\frac{2\pi ij}{n}}\sqrt[n]{z}) \tag{81}$$

whenever $|z|<1$.

Remark 2. The choice of branch for $\sqrt[n]{\cdot}$ does not matter, so long as the choice is consistent across the expression for $G(z)$. To see this more clearly, rewrite

$$G(z)=-\frac{m}{n}\sum_{j=0}^{n-1}\left(e^{-\frac{2\pi ij}{n}}\frac{1}{\sqrt[n]{z}}\right)^m\log\left(1-e^{\frac{2\pi ij}{n}}\sqrt[n]{z}\right). \tag{82}$$

This sum is symmetric over the n^{th} roots of z. Any branch of $\sqrt[n]{\cdot}$ must map an input z to one of the n possible roots ω of $\omega^n=z$. The symmetry in (82) implies that, no matter the branch chosen, this sum will always have the same terms.

From (70) we know that $G(z)={}_2F_1\left(1,\frac{m}{n};1+\frac{m}{n};z\right)$, and hence (81) allows us to conclude that for $\beta=\frac{m}{n}\in\mathbb{Q}\setminus\mathbb{Z}$ with $m\in\mathbb{Z}$, $n\in\mathbb{N}$,

$${}_2F_1\left(1,\frac{m}{n};1+\frac{m}{n};z\right)=-\frac{m}{n}z^{-\frac{m}{n}}\sum_{j=0}^{n-1}e^{-\frac{2\pi ijm}{n}}\log(1-e^{\frac{2\pi ij}{n}}\sqrt[n]{z}) \tag{83}$$

for all $|z|<1$. Finally, when $|\alpha|<1$ we have $|\alpha e^{-i\theta}|<1$, so substituting $z=\alpha e^{-i\theta}$ into (83) proves the first conclusion of Corollary 1. Similarly, when $|\alpha|>1$, we have that $|\frac{1}{\alpha}e^{i\theta}|<1$, and hence substituting $z=\alpha^{-1}e^{i\theta}$ into (83) proves the second conclusion. $\square$

6 Differential Equation

A key feature of the hypergeometric equation

$$z(1-z)\frac{d^2F}{dz^2}+(c-(a+b+1)z)\frac{dF}{dz}-abF=0 \tag{84}$$

is its regular singular points, and it is well-known that they are $0,1,\infty$. Hence, one might wish to derive a second-order ordinary differential equation in the variable α for which $I(\alpha)=\int_{\partial\mathbb{D}} m_{\alpha,\beta,\theta}$ is a solution, and determine its regular singular points.

6.1 The case $|\alpha|>1$

For $|\alpha|>1$, $\beta\notin\mathbb{Z}_{\leq 0}$, the desired equation follows by relating (1) to the hypergeometric function $F(z)={}_2F_1(a,b,c;z)$, which is famously a solution of (84).
Let $f(z)={}_2F_1(1,\beta,1+\beta;z)$. Then f solves the equation

$$z(z-1)\frac{d^2f}{dz^2}+((1+\beta)-(2+\beta)z)\frac{df}{dz}-\beta f=0. \tag{85}$$

Consider the change in variables $\alpha=\frac{1}{z}e^{i\theta}$ (equiv. $z=\frac{1}{\alpha}e^{i\theta}$), and make the following necessary calculations:

$$\begin{aligned}
\frac{df}{dz}&=\frac{d\alpha}{dz}\frac{df}{d\alpha}=-\frac{1}{z^2}e^{i\theta}\frac{df}{d\alpha}=-\alpha^2e^{-i\theta}\frac{df}{d\alpha},\\
\frac{d^2f}{dz^2}&=\frac{d\alpha}{dz}\cdot\frac{d}{d\alpha}\frac{df}{dz}\\
&=-\alpha^2e^{-i\theta}\left(-\alpha^2e^{-i\theta}\frac{d^2f}{d\alpha^2}-2\alpha e^{-i\theta}\frac{df}{d\alpha}\right)\\
&=\alpha^4e^{-2i\theta}\frac{d^2f}{d\alpha^2}+2\alpha^3e^{-2i\theta}\frac{df}{d\alpha}.
\end{aligned}$$

By substituting into (85), notice that $f_*(\alpha):=f\left(\frac{1}{\alpha}e^{i\theta}\right)$ solves

$$\alpha^{-1}e^{i\theta}\left(\alpha^{-1}e^{i\theta}-1\right)\left(\alpha^4e^{-2i\theta}\frac{d^2f_*}{d\alpha^2}+2\alpha^3e^{-2i\theta}\frac{df_*}{d\alpha}\right)+\left((1+\beta)-(2+\beta)(\alpha^{-1}e^{i\theta})\right)\left(-\alpha^2e^{-i\theta}\frac{df_*}{d\alpha}\right)-\beta f_*=0,$$

which after some simplification becomes

$$p_2(\alpha)\frac{d^2f_*}{d\alpha^2}+p_1(\alpha)\frac{df_*}{d\alpha}-\beta f_*=0, \tag{86}$$

where $p_2(\alpha)=\alpha^2-\alpha^3e^{-i\theta}, p_1(\alpha)=\alpha(\beta+4)-\alpha^2(\beta+3)e^{-i\theta}$.
From Theorem 1, $f_*(\alpha)=1-kI(\alpha)$, with the abbreviation $k=\frac{\beta}{e^{i\beta\theta}(1-e^{-2\pi i\beta})}$, and we calculate the derivatives to be

$$\begin{aligned}
\frac{df_*}{d\alpha}&=\frac{df_*}{dI}\frac{dI}{d\alpha}=-k\frac{dI}{d\alpha},\\
\frac{d^2f_*}{d\alpha^2}&=-k\frac{d^2I}{d\alpha^2}.
\end{aligned}$$

Substitution into (86) yields that $I(\alpha)$ solves the equation

$$p_2(\alpha)\left(-k\frac{d^2I}{d\alpha^2}\right)+p_1(\alpha)\left(-k\frac{dI}{d\alpha}\right)-\beta\left(1-kI\right)=0,$$

or rather,

$$p_2(\alpha)\frac{d^2I}{d\alpha^2}+p_1(\alpha)\frac{dI}{d\alpha}-\beta I=e^{i\beta\theta}(e^{-2\pi i\beta}-1). \tag{87}$$

From (87) it is clear that the normalized coefficients $\frac{p_1(\alpha)}{p_2(\alpha)}\alpha$ and $\frac{-\beta}{p_2(\alpha)}\alpha^2$ are analytic in a neighborhood of 0. Similarly, $\frac{p_1(\alpha)}{p_2(\alpha)}(\alpha-e^{i\theta})$ and $\frac{-\beta}{p_2(\alpha)}(\alpha-e^{i\theta})^2$ are analytic in a neighborhood of $e^{i\theta}$. These coefficients have poles at 0 and $e^{i\theta}$, so these are regular singular points. To classify the point at infinity, let $x=1/\alpha$ and rewrite (87) in x. Akin to a previous change of variables, one has

$$\frac{dI}{d\alpha}=-x^2\frac{dI}{dx},$$
$$\frac{d^2I}{d\alpha^2}=x^4\frac{d^2I}{dx^2}+2x^3\frac{dI}{dx},$$

so that (87) becomes

$$p_2\left(\frac{1}{x}\right)\left(x^4\frac{d^2I}{dx^2}+2x^3\frac{dI}{dx}\right)+p_1\left(\frac{1}{x}\right)\left(-x^2\frac{dI}{dx}\right)-\beta I=e^{i\beta\theta}(e^{-2\pi i\beta}-1),$$

or equivalently,

$$q_2(x)\frac{d^2I}{dx^2}+q_1(x)\frac{dI}{dx}-\beta I=e^{i\beta\theta}(e^{-2\pi i\beta}-1), \tag{88}$$

where $q_2(x)=x^2-xe^{-i\theta}$, $q_1(x)=-x(\beta+2)+(\beta+1)e^{-i\theta}$. By a similar line of reasoning, the regular singular points of (88) are $x=0$ and $x=e^{-i\theta}$, so $\alpha=\infty$ and $\alpha=e^{i\theta}$ are both regular singular points of (87).

Thus equation (87), for which (1) is a solution, has precisely three regular singular points at $0,e^{i\theta},\infty$, reminiscent of (84). Any function satisfying a differential equation with three regular singular points may be expressed using the hypergeometric function, so this result supports the validity of the relationship derived.

6.2 The case $|\alpha|<1$

When $|\alpha|<1$, $\beta\notin\mathbb{Z}_{\geq 0}$, one can proceed exactly as §6.1 and make use of Theorem 1. Let $g(z)={}_2F_1(1,-\beta,1-\beta,z)$. Then g solves the equation

$$z(z-1)\frac{d^2g}{dz^2}+((1-\beta)-(2-\beta)z)\frac{dg}{dz}+\beta g=0.$$

The change of variables $\alpha=ze^{i\theta}$ gives that $g_*(\alpha):=g(\alpha e^{-i\theta})$ solves

$$r_2(\alpha)\frac{d^2g_*}{d\alpha^2}+r_1(\alpha)\frac{dg_*}{d\alpha}+\beta g_*=0, \tag{89}$$

where $r_2(\alpha)=\alpha^2-\alpha e^{i\theta}$, $r_1(\alpha)=(1-\beta)e^{i\theta}-(2-\beta)\alpha$. Theorem 1 gives $g_*(\alpha)=kI(\alpha)$, where again $k=\frac{\beta}{e^{i\beta\theta}(1-e^{-2\pi i\beta})}$. This scaling does not change the equation, so $I(\alpha)$ is also a solution of (89), with g_* replaced with I. Just as before, one reasons that $\alpha=0,e^{i\theta}$ are regular singular points of this equation. In the variable $x=\frac{1}{\alpha}$, the equation (89) written for I is

$$s_2(x)\frac{d^2I}{dx^2}+s_1(x)\frac{dI}{dx}+\beta I=0,$$

where $s_2(x) = x^2 - x^3e^{i\theta}, s_1(x) = \beta x - (1+\beta)e^{i\theta}x^2$, of which the regular singular points are $x = 0, e^{-i\theta}$. Finally, one concludes that the regular singular points of the hypergeometric-like differential equation that $I(\alpha)$ solves are $0, e^{i\theta}, \infty$.

7 Appendix

In order to use Lebesgue's Dominated Convergence Theorem, which is essential to the proofs in the paper, a short introduction to basic measure theory is needed. Careful treatment of the necessary measure theory topics is handled in Chapter 11 of Rudin's *Principles of Mathematical Analysis* (pgs. 300-315 of [3]). For more resources on the topic see the reviews in [4]. If the reader chooses to forego the short introduction to measure theory, it will at least help to understand the advantages of Lebesgue integration over the standard Riemann integration taught in introductory calculus.

For one, the Lebesgue integral extends to a much wider class of functions than the Riemann integral. A classic example is that of the Dirichlet function δ which takes value 1 on rationals and 0 on irrationals. This function is not Riemann integrable on the interval $[0,1]$, since no matter what partition P we pick for the interval, at least one irrational and one rational must lie in each interval of the partition. Thus for every partition P, $U(P,\delta) = 1$ and $L(P,\delta) = 0$, where U and L are the upper and lower sums of δ with partition P, respectively. This shows that δ is not Riemann integrable.

The Dirichlet function *is* Lebesgue integrable however, and its Lebesgue integral over the interval $[0,1]$ is simple to compute. Let λ be the Lebesgue measure on $\mathbb{R}$. By definition of the Lebesgue integral, we have

$$\int_{[0,1]} \delta \, d\lambda = 1 \cdot \lambda([0,1] \cap \mathbb{Q}) + 0 \cdot \lambda([0,1] \setminus \mathbb{Q}) = 0$$

since the rationals form a Lebesgue measure 0 subset of the reals. This example shows in particular that a "large" number of discontinuities (think uncountably many, as in the Dirichlet function) does not necessarily prevent a function from being Lebesgue integrable, while it does prevent it from being Riemann integrable.

Another limitation of Riemann integration is the difficulty in passing a limit under the integral sign. Given a sequence of Riemann-integrable functions $\{f_n\}$ which converge to some function f, it would be convenient if

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n\to\infty} f_n(x)\, dx.$$

Indeed, there are cases where this holds – for instance, if $\{f_n\}$ converges to a function f *uniformly* (rather than just pointwise) on the finite interval $[a,b]$. This fact can be seen as a consequence of Lebesgue's Dominated Convergence Theorem, again demonstrating its utility (see the comment under Exercise 11.6 in [3]). It is mainly for this reason we use the integral of Lebesgue rather than that of Riemann, since in the Lebesgue context it is much easier to justify switching the limit and integral.

The reader should also notice that we discuss measurability and integrability of functions with codomain $\mathbb{C}$ rather than $\mathbb{R}$ (see pg. 325 of [3]). Recall that $f : \mathbb{R} \to \mathbb{C}$ may be written in terms of two component functions $u, v : \mathbb{R} \to \mathbb{R}$

$$f(t) = u(t) + iv(t),$$

and the integral of f over $A \subseteq \mathbb{R}$ is aptly defined

$$\int_A f(t)\, d\lambda := \int_A u(t)\, d\lambda + i \int_A v(t)\, d\lambda.$$

It is natural then that the function $f : \mathbb{R} \to \mathbb{C}$ be integrable over A so long as u and v are. This also corroborates the definition that $f : \mathbb{R} \to \mathbb{C}$ is measurable if u and v are.

The need to discuss complex valued functions stems from our use of line integrals (pgs. 101-102 in [5]). Let $\Omega \subseteq \mathbb{C}$, and recall that the integral of the function $f : \Omega \to \mathbb{C}$ over a piecewise differentiable arc γ parameterized by $z : [a,b] \to \mathbb{C}$ is defined as

$$\int_\gamma f(z)\, dz := \int_a^b f(z(t)) z'(t)\, dt.$$

Defining $u_\gamma, v_\gamma : [a,b] \to \mathbb{R}$ such that

$$[(f \circ z)z'](t) = u_\gamma(t) + iv_\gamma(t),$$

the integral of f over γ can be written terms of two real integrals:

$$\int_\gamma f(z)\, dz = \int_a^b u_\gamma(t)\, dt + i \int_a^b v_\gamma(t)\, dt.$$

If f is continuous on the piecewise-continuously differentiable curve γ, then certainly $(f \circ z)z' : [a,b] \to \mathbb{C}$ is piecewise continuous and bounded. From here one concludes that u_γ, v_γ are also piecewise continuous, bounded functions; the component functions are in fact Riemann integrable. It is known that if a function $g : \mathbb{R} \to \mathbb{R}$ is Riemann integrable on $[a,b]$, then g is also Lebesgue integrable on $[a,b]$ and the two separate notions of integration yield the exact same result (Theorem 11.33 [3]). In this paper, every claim of integrability is justified through this sense.

Now we state Lebesgue's Dominated Convergence Theorem (Theorem 11.32 in [3]).

Theorem 2. *Suppose A is a measurable set with respect to some measure μ, and let $\{f_n\}$ be a sequence of measurable functions (with respect to the same measure) such that*

$$f_n(x) \to f(x) \quad \text{as} \quad n \to \infty \quad \forall x \in A.$$

If there exists a μ-integrable function g on A such that

$$|f_n(x)| \leq g(x) \quad \forall n \in \mathbb{N}, \forall x \in A,$$

then

$$\lim_{n \to \infty} \int_A f_n\, d\mu = \int_A f\, d\mu.$$

Remark 3. In our case, the sets we integrate over are always intervals. Since we choose $\mu = \lambda$ when applying Theorem 2, and since intervals are Lebesgue measurable, the first condition of Theorem 2 holds easily.

In the mainline discussion, we sweep under the rug the application of Theorem 2 to separate real and imaginary components. However it should be clear that the arguments laid out there justify the separate applications — the following points support this:

(a) a sequence of functions $f_n : \mathbb{R} \to \mathbb{C}$ are measurable in our sense, then by definition one had to have shown the component sequences u_n, v_n are measurable.

(b) $f_n \to f$ pointwise, then also the components converge $u_n \to u$ and $v_n \to v$ pointwise.

(c) $|f_n| \leq g$, then $|u_n|, |v_n| \leq |f_n| \leq g$.

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