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Matricial Frameworks for the Mandelbrot and Filled Julia Sets

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Abstract

Both the Mandelbrot set and filled Julia sets are subsets in the complex plane derived by studying iterations of complex polynomials. We develop a matricial framework to establish an alternate form of iteration by complex polynomials using a sequence of affine transformations. Using this framework, we are able to check membership in a filled Julia set and the Mandelbrot set by studying boundedness of sequences of matrices. Specifically, we show that a complex number belongs to the Mandelbrot set if and only if a particular sequence of matrices is bounded in the operator norm, and a complex number belongs to a filled Julia set if and only if a particular sequence of matrices is bounded in operator norm.

1 Introduction

The complex plane (denoted $\mathbb{C}$) is comprised of values $x+yi$, where x and y belong to the real numbers (denoted $\mathbb{R}$) and i is the imaginary number $\sqrt{-1}$. We call x the real part of $x+yi$ and y the imaginary part of $x+yi$. When we don't need to display the real and imaginary parts of a complex number, we simply denote it by a single variable, like z, c, or w.

1.1 Visualizations of the Complex Plane

To the unfamiliar eye, the complex plane looks no different than the familiar xy-coordinate plane, denoted $\mathbb{R}^2$. Indeed, a complex number $x+yi$ in $\mathbb{C}$ is spatially in the exact same location as the coordinate pair (x,y) in $\mathbb{R}^2$.

The perspective of vector geometry further supports the notion that these two spaces are not distinct. To do this, we can view a complex number of the from $x+yi$ as a vector with head at the origin which points a value of y in the direction of the vertical axis (imaginary values) and a value of x in the horizontal axis. Simply put, this representation allows for the complex plane to be viewed as a two-dimensional real vector space. Now, let $x_1, y_1, x_2, y_2 \in \mathbb{R}$. If we consider $x_1+y_1i, x_2+y_2i \in \mathbb{C}$ and $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$ as 2-dimensional vectors emanating from their respective origins (in $\mathbb{C}$ and $\mathbb{R}^2$), vector addition will yield outputs also in the exact same locations:

$$\text{In } \mathbb{R}^2: \qquad (x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2).$$

$$\text{In } \mathbb{C}: \qquad (x_1+y_1i)+(x_2+y_2i) = x_1+x_2+(y_1+y_2)i.$$

In contrast, there is a natural multiplication on $\mathbb{C}$ that $\mathbb{R}^2$ does not have.

$$\begin{aligned}\text{In } \mathbb{C}: (x_1+y_1i)*(x_2+y_2i) &= x_2x_2+x_1y_2i+x_2y_1i+y_1y_2i^2 \\ &= x_2x_2+x_1y_2i+x_2y_1i+y_1y_2(-1) \\ &= x_1x_2-y_1y_2+(x_1y_2+x_2y_1)i.\end{aligned}$$

Clearly, coordinate-version of the complex product, $(x_1x_2-y_1y_2, x_1y_2+x_2y_1)$ is not equal to the $\mathbb{R}^2$ product (x_1x_2, y_1y_2). An interesting way to interpret the strange multiplication on $\mathbb{C}$, is that we can also think of elements of $\mathbb{C}$ not only as 2-dimensional vectors, but also as 2×2-matrices with real entries:

$$x+yi \mapsto \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

Matrix multiplication appropriately implements the multiplication on $\mathbb{C}$. Observe:

$$\begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}\begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix} = \begin{bmatrix} x_1x_2 - y_1y_2 & x_1y_2 + x_2y_1 \\ -x_1y_2 - x_2y_1 & x_1x_2 - y_1y_2 \end{bmatrix}$$

Notice that the $(1,1)$-entry of the matrix product above is $x_1x_2 - y_1y_2$, which is the real part of the complex product $x_1x_2 - y_1y_2 + (x_1y_2 + x_2y_1)i$, and the $(1,2)$-entry of the matrix product above is $x_1y_2 + x_2y_1$, which is the imaginary part of the complex product $x_1x_2 - y_1y_2 + (x_1y_2 + x_2y_1)i$.

Functions like $f(z) = z^2 + i$, or more generally, $f(z) = z^2 + c$ for some complex constant c, play a central role in the definitions of the Mandelbrot and filled Julia sets. Throughout the paper, we study particular properties of functions like f, and these properties depend deeply on the constant c. So, instead of denoting the map $z \mapsto z^2 + c$ by just "f," we will denote it by f_c to emphasize the dependence on c.

The function $f_c(z) = z^2 + c$ does two sequential processes to an input w:

$$w \quad \xrightarrow{(1)} \quad w^2 \quad \xrightarrow{(2)} \quad w^2 + c$$

The first process is most easily visualized in yet another representation of complex numbers, called *polar form*. In this perspective, $w = re^{i\theta}$, where $r, \theta \in \mathbb{R}^+$, which is graphed by going a distance r along the positive real axis and then rotating that point an angle of θ counterclockwise. In polar form, $w \mapsto w^2$ is equivalent to $re^{i\theta} \mapsto (re^{i\theta})^2 = r^2e^{i(2\theta)}$. In the latter, the distance r that w was from the origin is squared, and the angle θ that w was from the origin is doubled. The second process sending w^2 to $w^2 + c$ is most easily visualized in the 2-dimensional vector representation of $\mathbb{C}$ as translation of the point w^2 along the vector representation of c.

We want to emphasize here that **neither of these two processes are linear**. Indeed, squaring a number is a *quadratic* transformation, while translating a number is an *affine* transformation. The main results of this paper nonetheless reframe the iterative process applying f_c to w into a *linear* transformation. Our goal in pursuing this unnatural framework is to leverage the robust tools of linear algebra.

1.2 Filled Julia Sets and the Mandelbrot Set

While the debate of who discovered Julia sets and, subsequently, the Mandelbrot set (named after mathematician Benoit Mandelbrot), is ongoing within the mathematical research community [4], there is no debate on the important role Julia sets and the Mandelbrot set play in seemingly disparate scientific fields. Within mathematics, the Mandelbrot set and most Julia sets are examples of *fractals*, which are geometric subsets of the complex plane that have properties similar to the frozen crystals on a snowflake when you zoom in under a microscope. These subsets arise not only in fractal geometry, but also in computer graphics, control theory [11], robotics, and even various methods of encryption. For example, strong QR codes often have a fractal embedded in them, such as a Julia set [7]. The strength of using a Julia set as means of creating a "password" comes from the complexity of drawing the Julia set itself.

Classically, the Mandelbrot set and filled Julia sets are constructed by repeatedly iterating a complex polynomial of the form $f_c(z) = z^2 + c$ at points in the complex plane. Each point iterated is called a *seed*. A seed belongs to a filled Julia set and/or the Mandelbrot set depending on the long run behavior of repeated iterations by f_c – in particular, whether or not repeated iterations form a bounded sequence.

The largest road block in studying these fractals is the computational complexity of this process of iteration. Below is an image of a Julia set created in C^{++} whose pixels are colored by how quickly they escape boundedness. The light green points escape very slowly and border the points inside the filled Julia set who do not escape. The dark points on the outside escape very quickly.

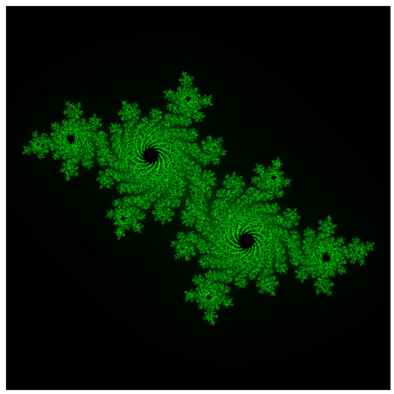

Figure 1.1: Julia Image for c = -.4 + -.59i

2 Preliminaries

For each $c \in \mathbb{C}$, define $f_c : \mathbb{C} \to \mathbb{C}$ by $f_c(z) = z^2 + c$.

2.1 Orbits and filled Julia sets

For $c \in \mathbb{C}$, the *orbit* of a complex number w under iterations of f_c, denoted $\mathscr{O}_c(w)$, is the sequence $\mathscr{O}_c(w) = \left\{ f_c^{(k)}(w) \right\}_{k=0}^{\infty}$, where $f_c^{(k)}(z)$ is the composition of k copies of f_c for $k \in \mathbb{N}$ and $f_c^{(0)}(z) = z$. We call w the *seed* of the orbit and c the *root* of the orbit. We say the orbit $\mathscr{O}_c(w)$ is *bounded* if there exists an $R > 0$ such that $\mathscr{O}_c(w)$ is a subset of $D(0,R) = \{z \in \mathbb{C} : |z| < R\}$. That is, an orbit is bounded if there exists an open disc centered at the origin in $\mathbb{C}$ that contains all elements of $\mathscr{O}_c(w)$.

Each orbit contains infinitely many complex numbers, and some orbits display their behavior relatively quickly (in the first few values), while other orbits take much longer (a lot of values) to show their true nature. Indeed, an orbit can produce two behaviors: the first being trending towards infinity and the second being bounded. When an orbit $\mathscr{O}_c(w)$ is bounded, it can have multiple sub behaviors–it could stay in a closed distance from a specific point, or it can produce a periodic orbit within a set of finitely many points.

Definition 2.1. *For $c \in \mathbb{C}$, the filled Julia set for c, denoted $\mathscr{J}_c$, is the set*

$$\mathscr{J}_c = \{w \in \mathbb{C} : \mathscr{O}_c(w) \text{ is bounded}\}.$$

Definition 2.2. *The Mandelbrot set, denoted $\mathfrak{M}$, is the set of all $c \in \mathbb{C}$ such that $\mathscr{J}_c$ is connected, i.e., $\mathscr{J}_c$ cannot be separated by two or more disjoint open subsets of $\mathbb{C}$.*

We will use an equivalent definition for this paper, which follows from Theorem 2.3 – see Proposition 3.1 in [2].

Theorem 2.3. *A complex number c is in the Mandelbrot set $\mathfrak{M}$ if and only if $0 \in \mathscr{J}_c$.*

For $c \in \mathbb{C}$, it is easy to see that $\mathscr{O}_c(0)$ is bounded if and only if $\mathscr{O}_c(c) = \mathscr{O}_c(0) \setminus \{0\}$ is bounded, which yields the following definition. The *Mandelbrot set* is the set of all complex numbers c such that $\mathscr{O}_c(c)$ is bounded, or equivalently, $c \in \mathscr{J}_c$, i.e., $\mathfrak{M} = \{c \in \mathbb{C} : c \in \mathscr{J}_c\}$.

2.2 Norms on Matrix Algebras

In our work, we will be interested determining the convergence of sequences of matrices. Given a matrix norm $\|\cdot\|$ on $M_n(\mathbb{R})$, a sequence $\{A_k\}_{n=1}^{\infty}$ of matrices in $M_n(\mathbb{R})$ converges to a matrix A with respect to $\|\cdot\|$ if

$$\lim_{k\to\infty} \|A_k - A\| = 0.$$

The two primary matrix norms we use are the *operator* and *Frobenius* norms, denoted $\|\cdot\|_{op}$ and $\|\cdot\|_F$, respectively. Given a matrix $A \in M_n(\mathbb{R})$, the *operator norm* of A is

$$\|A\|_{op} = \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

and the *Frobenius norm* of A is

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \left|A_{ij}\right|^2},$$

where A_{ij} is the ij-entry of A. It is well-known that the operator norm of a matrix A always equals its largest *singular value*, where the *singular values* of A are defined to be the square root of the eigenvalues of the matrix A^TA, and the Frobenius norm of A is always an upper bound for the operator norm of A, i.e., $\|A\|_{op} \leq \|A\|_F$.

2.3 Real and Imaginary Parts of an Iteration

Let $z = x + yi$ and $c = a + bi$. We denote the first iteration of z under f_c by z_1 – that is, $z_1 := f_c(z) = z^2 + c$. Observe

$$\begin{aligned} z_1 &= z^2 + c \\ &= (x+yi)^2 + (a+bi) \\ &= x^2 + 2xyi - y^2 + a + bi \\ &= \underbrace{(x^2 - y^2 + a)}_{x_1} + \underbrace{(2xy + b)}_{y_1} i. \end{aligned}$$

When we compute z_1 whilst keeping track of the real and imaginary parts of the seed z and root c, we are able to give a precise formula for the real and imaginary parts of $z_1 = x_1 + y_1 i$:

$$x_1 = x^2 - y^2 + a \qquad \text{and} \qquad y_1 = 2xy + b.$$

In fact, for all $k \in \mathbb{N}$, the real and imaginary parts of $z_k = x_k + y_k i$ are given recursively by the formulae

$$x_{k+1} = x_k^2 - y_k^2 + a \qquad \text{and} \qquad y_{k+1} = 2x_k y_k + b. \tag{1}$$

2.4 Affine Transformations

Our work aims to build a matricial framework for determining membership in the Mandelbrot and filled Julia sets. We do so by viewing $\mathbb{C}$ as a 2-dimensional real vector space and each consecutive element of an orbit under f_c as an *affine transformation* of the previous element. Thus, each orbit yields a sequence of *affine transformation matrices* from the algebra of 3×3 matrices with real entries, denoted $M_3(\mathbb{R})$.

Viewing $\mathbb{R}^2$ as a 2-dimensional real vector space, an *affine transformation* $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a transformation that can be decomposed as $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, where A is a linear transformation and $\mathbf{b}$ is a vector along which you subsequently translate the vector $A\mathbf{x}$. In our work, we prefer to work solely with linear transformations, which requires us to utilize what are called *homogeneous coordinates.* Each vector $\mathbf{x}$ in $\mathbb{R}^2$ can be written in homogeneous coordinates as $\begin{bmatrix}\mathbf{x} & 1\end{bmatrix}^T$, and as such, an affine transformation T defined above can be realized as the projection of a matrix transformation $\tilde{T}$ on $\mathbb{R}^3$ given by

$$\tilde{T}\left(\begin{bmatrix}\mathbf{x}\\ 1\end{bmatrix}\right) = \begin{bmatrix}A & \mathbf{b}\\ \mathbf{0}^T & 1\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ 1\end{bmatrix} = \begin{bmatrix}A\mathbf{x}+\mathbf{b}\\ 1\end{bmatrix},$$

where the standard matrix for $\tilde{T}$ is called the *affine transformation matrix for T*.

3 Matricial Framework for the Mandelbrot Set

As $\mathbb{R}$-vector spaces, $\mathbb{R}^2$ and $\mathbb{C}$ are isomorphic: given $z = x + yi \in \mathbb{C}$, identify z with the vector $\begin{bmatrix}x\\ y\end{bmatrix} \in \mathbb{R}^2$. Fix $c = a + bi \in \mathbb{C}$. Then the complex number $f_c(z)$ would be identified with the corresponding vector in $\mathbb{R}^2$:

$$\begin{bmatrix}x\\ y\end{bmatrix} \sim z \mapsto f_c(z) \sim \begin{bmatrix}x^2 - y^2 + a\\ 2xy + b\end{bmatrix}.$$

Not surprisingly, this mapping on $\mathbb{R}^2$ is not linear, and thus cannot be implemented by matrix multiplication of a single 2×2-matrix with real entries. Instead, notice:

$$\begin{bmatrix}x^2 - y^2 + a\\ 2xy + b\end{bmatrix} = \begin{bmatrix}x & -y\\ 2y & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}a\\ b\end{bmatrix}.$$

When the input z is the fixed complex number c, as is of interest when determining if c belongs to the Mandelbrot set, the transformation $c \mapsto f_c(c)$ is implemented in $\mathbb{R}^2$ by first applying a linear transformation to $\begin{bmatrix} a \\ b \end{bmatrix}$ and then translating by $\begin{bmatrix} a \\ b \end{bmatrix}$:

$$\begin{bmatrix} a \\ b \end{bmatrix} \sim c \mapsto f_c(c) \sim \begin{bmatrix} a^2 - b^2 + a \\ 2ab + b \end{bmatrix} = \begin{bmatrix} a & -b \\ 2b & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}.$$

As described in subsection 5.1, this is an affine transformation, and thus can be realized as a linear transformation when we "lift" to $\mathbb{R}^3$:

$$\begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \sim c \mapsto f_c(c) \sim \begin{bmatrix} a^2 - b^2 + a \\ 2ab + b \\ 1 \end{bmatrix} = \begin{bmatrix} a & -b & a \\ 2b & 0 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$$

Recall that c belongs to the Mandelbrot set if and only if $\mathscr{O}_c(c) = \{f_c^{(k)}(c) : k \in \mathbb{N}\}$ is bounded. Set $a_0 = a$ and $b_0 = b$. For $k \in \mathbb{N}$, define a_k and b_k to be the real and imaginary parts of $f_c^{(k)}(c)$, respectively, i.e.,

$$a_{k+1} + b_{k+1} i = f_c^{(k+1)}(a + bi) = f_c(a_k + b_k i).$$

For each $k \in \mathbb{N}$, consider the 3×3 matrix

$$\begin{bmatrix} a_k & -b_k & a \\ 2b_k & 0 & b \\ 0 & 0 & 1 \end{bmatrix},$$

and observe that

$$\begin{bmatrix} a_k & -b_k & a \\ 2b_k & 0 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_k \\ b_k \\ 1 \end{bmatrix} = \begin{bmatrix} a_k^2 - b_k^2 + a \\ 2a_k b_k + b \\ 1 \end{bmatrix}$$

By Equation 1, this means that for each $k \in \mathbb{N}$ we have

$$\begin{bmatrix} a_{k+1} \\ b_{k+1} \\ 1 \end{bmatrix} = \begin{bmatrix} a_k & -b_k & a \\ 2b_k & 0 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_k \\ b_k \\ 1 \end{bmatrix}. \tag{2}$$

Equation 2 shows that the orbit $\mathscr{O}_c(c)$ can be generated by recursively applying affine transformations to the real and imaginary parts of the previous image. In particular, for each $k \in \mathbb{N}$, define $A_k : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$A_k(\mathbf{x}) = \begin{bmatrix} a_{k-1} & -b_{k-1} \\ 2b_{k-1} & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} a \\ b \end{bmatrix}.$$

Then Equation 2 shows that the k^{th} iteration (written as an $\mathbb{R}^2$ vector in homogeneous coordinates) of f_c can be obtained by multiplying the $(k-1)^{\text{th}}$ iteration (written as an $\mathbb{R}^2$ vector in homogeneous coordinates) by the affine transformation matrix for A_{k-1}.

Define $M : \mathbb{R}^2 \to M_3(\mathbb{R})$ by $M(x,y) = \begin{bmatrix} x & -y & 0 \\ 2y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Hence, we can rewrite Equation 2 as

$$\begin{bmatrix} a_k \\ b_k \\ 1 \end{bmatrix} = \left(M(a_{k-1}, b_{k-1}) + \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} a_{k-1} \\ b_{k-1} \\ 1 \end{bmatrix}. \tag{3}$$

We continue by studying properties of the sequence of matrices $\{M(a_k, b_k)\}_{k=0}^{\infty}$, which arise from the generation of $\mathscr{O}_c(c)$.

3.1 Properties of $M(x,y)$

Our work was motivated by determining if properties of $\{M(a_k, b_k)\}_{k=0}^{\infty}$ tells us anything about whether or not $c = a + bi$ is in the Mandelbrot set. In this section, we outline the spectral theory of $M(x,y)$.

Let $x, y \in \mathbb{R}$ be given. A quick computation shows that the eigenvalues for $M(x,y)$ are

$$\lambda(x,y) = \frac{x \pm \sqrt{x^2 - 8y^2}}{2} \quad \text{and} \quad \lambda = 0.$$

Hence, we can define functions $\lambda_+, \lambda_- : \mathbb{R}^2 \to \mathbb{C}$ by

$$\lambda_+(x,y) = \frac{x + \sqrt{x^2 - 8y^2}}{2} \qquad \text{and} \qquad \lambda_-(x,y) = \frac{x - \sqrt{x^2 - 8y^2}}{2}.$$

If $y \neq 0$, i.e., $x + yi \notin \mathbb{R}$, it's easy to check that the eigenspaces corresponding to $\lambda_\pm(a,b)$ are

$$E_{\lambda_+(x,y)} = \operatorname{span}\left\{ \begin{bmatrix} \frac{x+\sqrt{x^2-8y^2}}{4y} \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad E_{\lambda_-(x,y)} = \operatorname{span}\left\{ \begin{bmatrix} \frac{x-\sqrt{x^2-8y^2}}{4y} \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Likewise, if $y = 0$, i.e., $x + yi \in \mathbb{R}$, then $M(x,0)$ is diagonal with eigenspaces $E_x = \operatorname{span}\{\mathbf{e}_1\}$ and $E_0 = \operatorname{span}\{\mathbf{e}_2, \mathbf{e}_3\}$, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis for $\mathbb{R}^3$.

Lemma 3.1. *$M(x,y)$ has distinct eigenvalues if and only if $x \neq \pm 2\sqrt{2}y$.*

Proof. A quick computation yields $\lambda_+(x,y) = \lambda_-(x,y)$ if and only if $x = \pm 2\sqrt{2}y$, which proves the lemma. □

Proposition 3.2. *If $(x,y) \neq (0,0)$, then $M(x,y)$ is diagonalizable if and only if $x \neq \pm 2\sqrt{2}y$.*

Proof. Note that $M(0,0)$ is the 3×3 zero matrix, which is trivially diagonalizable. Suppose $(x,y) \neq (0,0)$. If $x \neq \pm 2\sqrt{2}y$, then $M(x,y)$ is diagonalizable since it has 3 distinct eigenvalues by the previous lemma.

It remains to show that if $M(x,y)$ is diagonalizable, then $x \neq \pm 2\sqrt{2}y$. We prove the contrapositive. If $x = \pm 2\sqrt{2}y$, then

$$\lambda_+(x,y) = \lambda_-(x,y)$$

by the previous lemma. Hence, we have $E_{\lambda_+(x,y)} = E_{\lambda_-(x,y)}$ is 1-dimensional, which implies $M(x,y)$ is not diagonalizable. □

Proposition 3.3. *$M(x,y)$ is singular for all $x,y \in \mathbb{R}$.*

Proof. Since $\lambda = 0$ is an eigenvalue, $M(x,y)$ is not invertible. □

It can be computed directly that the nonzero singular values of $M(x,y)$ are

$$\sigma_-(x,y) := \frac{1}{\sqrt{2}}\left(x^2+5y^2-\sqrt{x^4+10x^2y^2+9y^4}\right)^{1/2}$$
$$\sigma_+(x,y) := \frac{1}{\sqrt{2}}\left(x^2+5y^2+\sqrt{x^4+10x^2y^2+9y^4}\right)^{1/2}$$

Proposition 3.4. *The operator norm of $M(x,y)$ is $\sigma_+(x,y)$.*

Proof. The maximum singular value of $M(x,y)$ is $\sigma_+(x,y)$. □

3.2 Boundedness and Convergence of Iterations

We show that boundedness of an orbit $\mathscr{O}_{a+bi}(a+bi)$ is equivalent to boundedness of the sequence of matrices $\{M(a_k,b_k)\}_{k=0}^{\infty}$ with respect to the operator norm.

Proposition 3.5. *Let $c = a+bi$ and $A+Bi \in \mathbb{C}$ be given. Then $\{f_c^{(k)}(c)\}_{k=0}^{\infty}$ converges to $A+Bi$ if and only if $\{M(a_k,b_k)\}_{k=0}^{\infty}$ converges to $M(A,B)$ in operator (and Frobenius) norm.*

Proof. Observe that

$$M(a_k,b_k)-M(A,B) = \begin{bmatrix} a_k-A & -(b_k-B) & 0 \\ 2(b_k-B) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, we have $\|M(a_k,b_k)-M(A,B)\|_F^2 = (a_k-A)^2+5(b_k-B)^2$. Since $(a_k-A)^2, (b_k-B)^2 \geq 0$ for all $k \in \mathbb{N}$, it follows that $\lim_{k\to\infty}\|M(a_k,b_k)-M(A,B)\|_F = 0$ if and only if

$$\lim_{k\to\infty}\left|f_c^{(k)}(c)-(A+Bi)\right| = \lim_{k\to\infty}\sqrt{(a_k-A)^2+(b_k-B)^2} = 0.$$

Therefore, $\{f_c^{(k)}(c)\}_{k=0}^{\infty}$ converges to $A+Bi$ if and only if $\{M(a_k,b_k)\}_{k=0}^{\infty}$ converges to $M(A,B)$ in Frobenius norm.

Since the operator and Frobenius norms are equivalent on $M_3(\mathbb{R})$, it follows that $\{M(a_k,b_k)\}_{k=0}^{\infty}$ converges to $M(A,B)$ in operator norm if and only if $\{M(a_k,b_k)\}_{k=0}^{\infty}$ converges to $M(A,B)$ in Frobenius norm, which completes the proof.

□

The following result justifies the study of our matricial framework in the context of the Mandelbrot set.

Theorem 3.6. *A complex number $c = a+bi$ is in the Mandelbrot set if and only if $\{M(a_k,b_k)\}_{k=0}^{\infty}$ is uniformly bounded in operator norm.*

Proof. Suppose $c = a + bi$ is in the Mandelbrot set. Then there exists an $R > 0$ such that $\left\| \begin{bmatrix} a_k \\ b_k \end{bmatrix} \right\|^2 = \left| f_c^{(k)}(c) \right|^2 < R$ for all $k \in \mathbb{N} \cup \{0\}$. It follows that for each $k \in \mathbb{N} \cup \{0\}$ we have

$$\|M(a_k, b_k)\|_F^2 = a_k^2 + 5b_k^2 \leq 5a_k^2 + 5b_k^2 < 5R.$$

Thus, $\sqrt{5R}$ is an upper bound for $\left\{ \|M(a_k, b_k)\|_{op} \right\}_{k=0}^{\infty}$ since the operator norm is bounded above by the Frobenius norm.

Conversely, suppose that $\{M(a_k, b_k)\}_{k=0}^{\infty}$ is bounded in operator norm. Then there exists a $C > 0$ such that $\|M(a_k, b_k)\|_{op} < C$ for all $k \in \mathbb{N} \cup \{0\}$. Hence, for each $k \in \mathbb{N} \cup \{0\}$ the definition of the operator norm yields

$$C > \|M(a_k, b_k)\|_{op} \geq \|M(b_k, a_k)\mathbf{e}_1\| = \left\| \begin{bmatrix} a_k \\ 2b_k \\ 0 \end{bmatrix} \right\| \geq \left\| \begin{bmatrix} a_k \\ b_k \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} a_k \\ b_k \end{bmatrix} \right\|,$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis for $\mathbb{R}^3$. Thus, $\left\{ \left\| \begin{bmatrix} a_k \\ b_k \end{bmatrix} \right\| \right\}_{k=0}^{\infty}$ is bounded, which implies $c = a + bi$ is in the Mandelbrot set. □

As a consequence of Theorem 3.6, we obtain a bound on the sequence of eigenvalues arising from the sequence of affine transformation matrices implementing f_c.

Corollary 3.7. If $c = a + bi$ is in the Mandelbrot set, then $\{\lambda_{\pm}(a_k, b_k)\}_{k=0}^{\infty}$ is bounded.

Proof. The spectral radius of a matrix is bounded above by its operator norm. □

4 Matricial Framework for Filled Julia Sets

Definition 4.1. *Given $a, b, x, y \in \mathbb{R}$, let $c := a + bi$ and $z := x + yi$. Define*

$$J(c,z) := \begin{bmatrix} x & -y & a \\ 2y & 0 & b \\ 0 & 0 & 1 \end{bmatrix}$$

Note that $J(c,z)$ can be decomposed similar to $M(x,y)$:

$$J(c,z) = \begin{bmatrix} x & -y & a \\ 2y & 0 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x & -y & 0 \\ 2y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 1 \end{bmatrix} = M(x,y) + T_{\mathbf{c},1}$$

where $T_{\mathbf{c},1}$ denotes translation of a point in $\mathbb{R}^3$ along the vector $\begin{bmatrix} a & b & 1 \end{bmatrix}^T$. Just like the action of f_c on a seed $z = x + yi$, the action of the linear operator $J(c,z)$ moves the vector $\begin{bmatrix} x & y & 1 \end{bmatrix}^T$ to $\begin{bmatrix} x_1 & y_1 & 1 \end{bmatrix}^T$ via two processes, (1) $M(x,y)$ and (2) $T_{\mathbf{c},1}$.

Below we collect some properties of the matrix $J(c,z)$.

- The eigenvalues for $J(c,z)$ are $\{1, \lambda_+(c,z), \lambda_-(c,z)\}$, where

$$\lambda_{\pm}(c,z) = \frac{x \pm \sqrt{x^2 - 8y^2}}{2}.$$

At first glance, the eigenvalues for $J(c,z)$ appear to be identical to those for $M(x,y)$. This is strange because $J(c,z)$ is a matrix which has nonzero entries in the $(1,3)$ and $(2,3)$-entries, while $M(x,y)$ is 0 in these two places. When one computes the eigenvalues of a matrix T by solving the equation $\det(T-\lambda I)=\mathbf{0}$, the algorithm can indeed ignore values along which one is not expanding. What is important to note, however, is that the eigenvalues for $J(c,z_k)$ *will* include information about c because z_k includes information about c.

- The corresponding eigenspaces for $J(c,z)$ are spanned by the eigenvectors

$$\mathbf{e_1}=\begin{bmatrix}\frac{by-a}{1-x+2y^2}\\ \frac{b(x-1)-2ay}{1-x+2y^2}\\ 1\end{bmatrix}\qquad \mathbf{e}_+=\begin{bmatrix}\frac{x-\sqrt{x^2-8y^2}}{4y}\\ 1\\ 0\end{bmatrix}\qquad \mathbf{e}_-=\begin{bmatrix}\frac{x+\sqrt{x^2-8y^2}}{4y}\\ 1\\ 0\end{bmatrix}$$

- Information about the real and imaginary parts of z can be stripped off the matrix $J(c,z)$: the trace of $J(c,z)$ is $x+1$ and the determinant of $J(c,z)$ is $2y^2$.

- $J(c,z)^TJ(c,z)=\begin{bmatrix}x^2+4y^2 & -xy & ax+2by\\ -xy & y^2 & -ay\\ ax+2by & -ay & a^2+b^2+1\end{bmatrix}$ is symmetric.

In applications of linear algebra to quantum physics, mathematicians are particularly interested in *commutation relations* between pairs of matrices. Recall that for $x,y\in\mathbb{R}$, multiplication is commutative: $xy=yx$. However, given two matrices $A,B\in M_n(\mathbb{R})$, matrix multiplication is not always commutative: $AB\neq BA$. A commutation relation is simply the formula one finds when computing $AB-BA$, which, when A and B do not commute, will yield a nonzero matrix. Because of its frequency in literature, mathematicians denote $AB-BA$ by $[A,B]$. In the following two lemmas, we determine precisely when two matrices $J(c,z)$ and $J(c,w)$, where z might be different from w, will commute, and we also show when $J(c_1,z)$ and $J(c_2,z)$ commute, where c_1 and c_2 may also be different.

Lemma 4.2. *Let $c,z,w\in\mathbb{C}$ and set $c=a+bi$, $z:=x+yi$ and $w:=f+gi$. If $a\neq 0$, $[J(c,z),J(c,w)]=\mathbf{0}$ if and only if $x=f$ and $y=g$. If $a=0$, $[J(c,z),J(c,w)]=\mathbf{0}$ if and only if $fy=xg$.*

Proof. Observe

$$\begin{aligned}[J(c,z),J(c,w)]=\mathbf{0}\iff &\begin{bmatrix}xf-2yg & -xg & ax-bf+a\\ 2fy & -2yg & 2ay+b\\ 0 & 0 & 1\end{bmatrix}\\ &-\begin{bmatrix}fx-2yg & -fy & af-bg+a\\ 2xg & -2yg & 2ag+b\\ 0 & 0 & 1\end{bmatrix}\\ &=\mathbf{0}\\ \iff &fy_1\overset{(1)}{=}xg\text{ and }ax-by\overset{(2)}{=}af-g\text{ and }ay\overset{(3)}{=}ag.\end{aligned}$$

Case 1 ($a\neq 0$): If $a\neq 0$, (3) is equivalent to $y=g$, and if $y,g\neq 0$, this implies $x\overset{(1)}{=}f$. If $y=0$ ($g=0$) without loss of generality) then (3) implies $g=0$ ($y=0$). Finally, this yields $ax \overset{(2)}{=} af$, which implies $x = f$ when $a \neq 0$.

Case 2 ($a = 0$): If $a = 0$, then $-by \overset{(2)}{=} -bg$. If $b = 0$, equations (2) and (3) are vacuous and $xg \overset{(1)}{=} fy$ is equivalent to $[J(c,z), J(c,w)] = \mathbf{0}$. If $b \neq 0$, we get $y \overset{(2)}{=} g$. If $y_1 \neq 0$ (or, equivalently, $g \neq 0$) then $x = f$. If $a = y = g = 0$, then x and f can be distinct. □

Lemma 4.3. *Let $c_1, c_2, z \in \mathbb{C}$, and set $c_1 := a_1 + b_1 i$, $c_2 := a_2 + b_2 i$, and $z = x + yi$. Then $[J(c_1,z), J(c_2,z)] = \mathbf{0}$ if and only if either $a = c$ or $a \neq c$, $y = \frac{b-d}{2(a-c)}$ and $x = \frac{-(b-d)^2}{2(a-c)^2} - 1$.*

Proof. One can easily show that $[J(c_1,z), J(c_2,z)] = \mathbf{0}$ if and only if

$$(a_1 - a_2)x + (b_1 - b_2)y + (a_1 - a_2) \overset{(1)}{=} 0 \qquad \text{and} \qquad 2ya_2 + b_1 \overset{(2)}{=} 2ya_1 + b_2.$$

When $a_1 \neq a_2$, (2) can be written as $y = \frac{b_1 - b_2}{2(a_1 - a_2)}$. One can then plug this expression in for y in (1) and simplify to get $x = \frac{-(b_1 - b_2)^2}{2(a_1 - a_2)^2} - 1$. When $a_1 = a_2$, (2) implies $b_1 = b_2$, in which case $c_1 = c_2$. □

The last result in this section is a generalization of the previous section's main theorem. Specifically, in the previous section's main theorem we showed that a complex number $c = a + bi$ belongs to the Mandelbrot set if and only if the sequence of matrices $\{M(a_k, b_k)\}_{k=0}^{\infty}$ is uniformly bounded in the operator norm. Equivalently, $c \in \mathscr{J}_c$ if and only if $\{M(a_k, b_k)\}_{k=0}^{\infty}$ is uniformly bounded in the operator norm. Below, we allow for the seed z to differ from c and prove an analogous result.

Theorem 4.4. *A complex number z belongs to $\mathscr{J}_c$ if and only if the set of matrices $\{J(c, z_k) : n \in \mathbb{N}\}$ is bounded in operator norm.*

Proof. Fix $c = a + bi$ and suppose $z = x + yi \in \mathscr{J}_c$. Then there exists an $R > 0$ such that $\left\| \begin{bmatrix} x_k \\ y_k \end{bmatrix} \right\|^2 = \left| f_c^{(k)}(z_k) \right|^2 < R$ for all $k \in \mathbb{N} \cup \{0\}$. It follows that for each $k \in \mathbb{N} \cup \{0\}$ we have

$$\|J(c, z_k)\|_F^2 = x_k^2 + 5y_k^2 + \left\| \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \right\|^2 \leq 5x_k^2 + 5y_k^2 + \left\| \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \right\|^2 < 5R + \left\| \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \right\|^2.$$

Thus, $\sqrt{5R + a^2 + b^2 + 1}$ is an upper bound for $\left\{ \|J(c, z_k)\|_{op} \right\}_{k=0}^{\infty}$ since the operator norm is bounded above by the Frobenius norm.

Conversely, suppose that $\{J(c, z_k)\}_{k=0}^{\infty}$ is bounded in operator norm. Then there exists a $C > 0$ such that $\|J(c, z_k)\|_{op} < C$ for all $k \in \mathbb{N} \cup \{0\}$. Hence, for each $k \in \mathbb{N} \cup \{0\}$ the definition of the operator norm yields

$$C > \|J(c, z_k)\|_{op} \geq \|J(c, z_k)\mathbf{e}_1\| = \left\| \begin{bmatrix} x_k \\ 2y_k \\ 0 \end{bmatrix} \right\| \geq \left\| \begin{bmatrix} x_k \\ y_k \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} x_k \\ y_k \end{bmatrix} \right\|,$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis for $\mathbb{R}^3$. Thus, $\left\{ \left\| \begin{bmatrix} x_k \\ y_k \end{bmatrix} \right\| \right\}_{k=0}^{\infty}$ is bounded, which implies $z = x + yi \in \mathscr{J}_c$.

□

5 Future Directions

5.1 Dynamical Systems and Markov Processes

In this section, we take $\mathbb{N}$ to denote $\{0,1,2,...\}$. Consider the set of countably infinite sequences of complex numbers:

$$\mathbb{C}^{\mathbb{N}} = \{(z_0, z_1, z_2, ...) : z_k \in \mathbb{C}\ \forall n \in \mathbb{N}\}.$$

We denote an element of $\mathbb{C}^{\mathbb{N}}$ by $(z_k)_{n=0}^{\infty}$. For any $k \in \mathbb{N}$, we write z_k to denote a single complex number within the sequence (z_k). Some of these sequences are related to the orbit of a seed $z \in \mathbb{C}$ under iterations of f_c for some root $c \in \mathbb{C}$. Indeed, if a sequence $(z_k)_{n=0}^{\infty} \in \mathbb{C}^{\mathbb{N}}$ satisfies $z_{n+1} = f_c(z_n)$ for all $n \in \mathbb{N}$, then $\mathscr{O}_c(z) = (z_k)_{n=0}^{\infty}$.

Example 5.1. Below we give some simple examples that show the presence of Julia set orbits inside the set $\mathbb{C}^{\mathbb{N}}$.

(i) If $c = 1$ and $z = 1$, then $\mathscr{O}_1(1) = (1, 2, 5, 26, ...)$.

(ii) If $c = i$ and $z = i$, then $\mathscr{O}_i(i) = (i, i-1, -i, i-1, ...)$.

(iii) If $c = i$ and $z = 2i$, then $\mathscr{O}_i(2i) = (2i, i-4, 15-7i, 176-209i, ...)$.

(iv) If $c = -1$ and $z = 1$, then $\mathscr{O}_{-1}(1) = (1, 0, -1, 0, ...)$.

To parallel our previous sections' work of re-framing iterations and orbits in terms of 2-dimensional real vectors, in this case, we would consider the set

$$(\mathbb{R}^2)^{\mathbb{N}} = \{(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2, ...) : \mathbf{z}_k \in \mathbb{R}^2\ \forall n \in \mathbb{N}\}.$$

Example 5.2. The analogue of the above examples for $\mathbb{C}^{\mathbb{N}}$ are easily translated into the $(\mathbb{R}^2)^{\mathbb{N}}$ picture.

(i) If $c = 1$ and $\mathbf{z} = \begin{bmatrix}1\\0\end{bmatrix}$, then $\mathscr{O}_1(\mathbf{z}) = \left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}5\\0\end{bmatrix}, \begin{bmatrix}26\\0\end{bmatrix}, ...\right)$.

(ii) If $c = i$ and $\mathbf{z} = \begin{bmatrix}0\\1\end{bmatrix}$, then $\mathscr{O}_i(\mathbf{z}) = \left(\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}, ...\right)$.

(iii) If $c = i$ and $\mathbf{z} = \begin{bmatrix}0\\2\end{bmatrix}$, then $\mathscr{O}_i(\mathbf{z}) = \left(\begin{bmatrix}0\\2\end{bmatrix}, \begin{bmatrix}-4\\1\end{bmatrix}, \begin{bmatrix}15\\-7\end{bmatrix}, \begin{bmatrix}176\\-209\end{bmatrix}, ...\right)$.

(iv) If $c = -1$ and $\mathbf{z} = \begin{bmatrix}1\\0\end{bmatrix}$, then $\mathscr{O}_{-1}(\mathbf{z}) = \left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, ...\right)$.

Just like in the $\mathbb{C}^{\mathbb{N}}$ setting, a sequence $(\mathbf{z}_k)_{n=0}^{\infty}$ of vectors in $(\mathbb{R}^2)^{\mathbb{N}}$ is the orbit $\mathscr{O}_{\mathbf{c}}(\mathbf{z})$ if

$$\text{for every } n \in \mathbb{N}, \qquad \mathbf{z}_{n+1} = J(\mathbf{z}_k, c)\mathbf{z}_k.$$

We can sort of think of the matrices $J(c, z_k)$ as detecting these orbits, but, of course, we have to check this equality holds at all $n \in \mathbb{N}$.

Given $w \in \mathbb{C}$, define a map $\mathsf{J}_{c,w} : (\mathbb{R}^2)^{\mathbb{N}} \to (\mathbb{R}^2)^{\mathbb{N}}$ by

$$\mathsf{J}_{c,w}(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2, ...) := (J(c,w_0)\mathbf{z}_0, J(c,w_1)\mathbf{z}_1, J(c,w_2)\mathbf{z}_2, ...).$$

When $z = w$, $\mathsf{J}_{c,w}$ is the *direct sum* of the matrices $J(c,w_k)$, commonly denoted $\oplus_{n=0}^{\infty} J(c,w_k)$. The notation and definition of direct sums will not be used in the rest of the paper, so we mention it only in case the reader is familiar with these operators.

Note that the action of $\mathsf{J}_{c,w}$ on the specific vector sequence $(\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, ...)$ just pushes each $\mathbf{w}_k$ to the left one position:

$$\mathsf{J}_{c,w}(\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, ...) = (J(c,w_0)\mathbf{w}_0, J(c,w_1)\mathbf{w}_1, J(c,w_2)\mathbf{w}_2, ...) = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, ...).$$

Since there's no place on the left for the $\vec{w}_0$ term to go, it just gets tossed off the left edge of the sequence; it walks the plank of the $\mathsf{J}_{c,w}$ pirate ship.

We will call $\mathsf{J}_{c,w}$ a "left shift operator," but we must note that, unlike classic left shift operators, $\mathsf{J}_{c,w}$ is only acting as a left shift on the subspace of vectors spanned by $(\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, ...)$ and images of $(\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, ...)$ under $\mathsf{J}_{c,w}$.

These operators have an interesting property. Recall that we found the eigenvalues for each $J(c,z_k)$ in a previous section: $\sigma(J(c,z_k)) = \{1, \lambda_+(c,z_k), \lambda_-(c,z_k)\}$. It's not hard to show that $\sigma(\mathsf{J}_{c,z})$ contains all of these eigenvalues, $\cup_{n=0}^{\infty} \sigma(J(c,z_k))$, and possibly more. We plan to study the operators $\mathsf{J}_{c,z}$ and their spectra in a future project and see how these relate to the filled Julia set $\mathscr{J}_c$.

6 Notation

- $c = a + bi$
- $z = x + yi$, $z_k = x_k + y_k i$
- $\mathscr{O}_c(z)$ - orbit of z under f_c
- $\mathscr{J}_c$ - filled Julia Set for c
- $\mathfrak{M}$ - Mandelbrot Set
- f_c - mapping of $z \mapsto z^2 + c$
- $M(x,y) = \begin{bmatrix} x & -y & 0 \\ 2y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- $J(c,z) = \begin{bmatrix} x & -y & a \\ 2y & 0 & b \\ 0 & 0 & 1 \end{bmatrix}$
- $\mathsf{J}_{c,z}$ - $\oplus_{k=0}^{\infty} J(c, z_k)$
- $M_n(\mathbb{R})$ - set of $n \times n$-matrices with real entries
- $\|\cdot\|_{op}$ - operator norm on $M_n(\mathbb{R})$
- $\|\cdot\|_F$ - Frobenius norm on $M_n(\mathbb{R})$
- $[A,B] = AB - BA$

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